MARKOV CHAIN MODELS TO ESTIMATE THE PREMIUM FOR EXTENDED HEDGE FUND LOCKUPS

by

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Abstract

A lockup period for investment in a hedge fund is a time period after making the investment during which the investor cannot freely redeem his investment. It is routine to have a one-year lockup period, but recently the requested lockup periods have grown longer. Assuming that the investor will rebalance his portfolio of hedge funds on a yearly basis, if permitted, we define the annual lockup premium as the difference between the expected return per year from an investment in a hedge fund with a nominal one-year lockup period and the expected return per year from an investment in a hedge fund with an extended lockup period, as a function of the length of that extended lockup period. We develop Markov chain models to estimate this lockup premium function. By solving systems of equations, we fit the Markov chain transition probabilities to three directly observable hedge fund performance measures: the persistence of return, the variance of return and the hedge-fund death rate. The model quantifies the way the lockup premium depends on these parameters.

1. Introduction

A *lockup period* for investment in a hedge fund is a time period after making the investment during which the investor cannot freely redeem his investment. It is routine to have a one-year lockup period, but recently the requested lockup periods have grown longer. It is reasonable for an investor in a hedge fund to expect compensation for the restricted investment opportunities imposed by an extended lockup condition, with the compensation increasing as the length of the lockup period increases. We regard that compensation as a *lockup premium*, and we ask: *What should that lockup premium be as a function of the length of the lockup period?*

We assume that the investor will rebalance his portfolio of hedge funds on a yearly basis, as permitted. Thus, we *define* the (annual) lockup premium as the difference between the expected return per year from an investment in a hedge fund with a nominal one-year lockup period and the expected return per year from an investment in a hedge fund with an extended lockup period, as a function of the length of that extended lockup period. Our definition accounts for lost gains due to rebalancing the investment portfolio in hedge funds, but not for other lost investment opportunities, so we provide a conservative estimate of the lockup premium.

With that definition specified, we develop mathematical models to estimate the lockup premium function as a function of key hedge-fund performance measures. Specifically, we develop both discrete-time and continuous-time Markov chain models for that purpose. Our main contribution is to take a modelling approach, but there also are significant challenges in deciding what modelling approach to use. We want a good model, one which is easy to understand, properly reflects the specific lockup conditions, has predictive power, can be effectively analyzed and can be fit to available data.

These requirements lead us to propose relatively simple three-state Markov chain models. By introducing models with relatively few parameters, we have fewer parameters to fit to data. We do not directly fit the natural model parameters, which are the Markov chain transition probabilities and the state-dependent returns, but instead we *indirectly fit the model* to more directly observable hedge fund performance measures, specifically, the persistence of return, the variance of return and the hedge-fund death rate. This indirect approach requires that we solve systems of equations to determine the required model parameters.

Hedge fund lockup is an instance of a classical liquidity problem. It is well known that an illiquid investment, which limits the holder's ability to redeem, usually offers higher return than a liquid investment, which can be redeemed freely. It is common to regard the spread (the difference in yields) as the *liquidity premium.* A popular example is a certificate of deposit (CD). Unlike a usual savings account, a CD restricts its owner from redeeming his money until the CD matures. As a consequence, banks offer higher interest rates for CD's than for regular savings accounts. Experience shows that the liquidity premium increases as the length of the time period increases, but increases more slowly as the time increases, so that overall the premium function is concave.

In practice, the liquidity premium for CD's is determined by market forces, and is usually taken as given. Stock option prices are also determined by market forces, but because of the complex dynamics, it has proven useful to have the Black-Scholes option pricing model and other related models to estimate what the price should be. In the same spirit, in our hedge fund context, we introduce models to help understand what the lockup premium should be.

While hedge fund lockup is a liquidity problem like a CD and many other liquidity problems, it has its own special character. There is a complication with hedge funds, because investors may actually have an early opportunity to redeem their investment. If the hedge fund performs very poorly, so that it ceases operating, then a significant portion of the investment is returned to investors, even if the lockup period has not expired. Thus, it might appear that there should be no liquidity problem at all, but the two extreme alternatives are not the only possibilities: Hedge fund performance may be weak, so that returns are low and future prospects are dim, even though the fund does not cease operating. The lockup prevents the investor from moving his investment away from such "sick" funds. This special way hedge fund lockup is treated makes the liquidity premium more complicated, providing motivation for more careful analysis.

Our proposed model directly responds to this special feature of hedge fund investments: *We consider three possible states for a hedge fund: good, sick and dead, and we assume that transitions among these states occur randomly according to a Markov chain.* In a dead state, the investor suffers a low return, but at the next yearly reinvestment opportunity the state changes to a good state, because the investor gets his money back and can invest in a new fund, which we take to be in the good state. (We assume that the investor gets all his investment back.) There is no extra penalty from the lockup associated with a dead fund, but there is from a sick fund. With only nominal one-year hedge fund lockup, we assume that investors will reinvest in a good fund at the next yearly reinvestment opportunity whenever any fund they have invested in becomes sick. In contrast, with the extended lockup period, no reinvestment is possible until the end of the lockup period. In the meantime, the sick fund may perform poorly, and produce low returns, but there also is a chance that it may rebound and become a good fund. Clearly, some care is needed to properly account for the various good and bad possibilities, which inevitably must be regarded as random events. The Markov chain models can capture the behavior described above, so should provide insight.

Of course, it remains to specify the three Markov chain states. We propose classifying the funds according to their returns. Specifically, we focus on the relative returns, represented as the percent difference from the average annual return for that category of funds. We say that a fund is in a: *good* state when its relative return is higher than U percent, *sick* state when its relative return is between L and U percent, and *dead* state when its relative return is less than L percent. We leave U and L as model parameters in general. Figure 1 illustrates an example of the distribution of annual returns of hedge fund with 4788 selected observations from 2001 to 2005 obtained from Tremont Advisory Shareholders Services (TASS) database. Tentative levels U and L show how states might be defined.

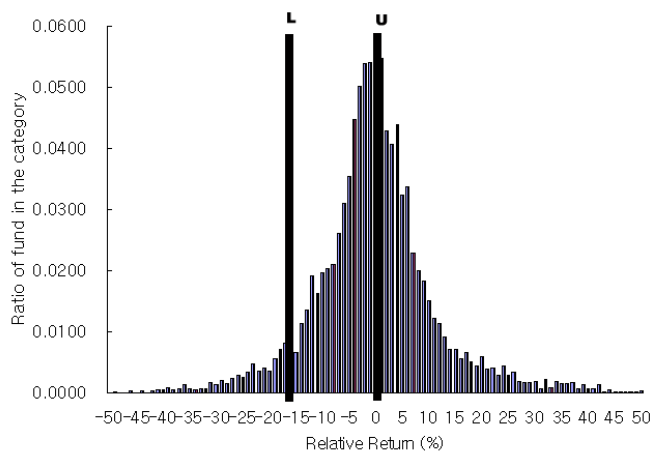


Figure 1: A distribution of hedge fund annual relative returns data with 4788 selected observations from 2001 to 2005. Tentative levels L and U divide the funds into the three states G, S and D.

A fundamental principal guides our analysis: the *persistence hypothesis*. We postulate that there is a persistence in hedge fund performance within a particular hedge fund strategy category: We assume that above-average funds will tend to continue doing well, while below-average funds will tend to continue faltering. A persistence of γ means that *for every* 1% *you earn above the average in the current year, you expect to earn* γ% *above the average in the next year.* We estimate the persistence by doing a regression analysis on the hedge-fund-return data from TASS, and find evidence to support the persistence hypothesis for some strategy categories of hedge funds.

As we will explain in §3, there is a consensus among researchers that there is persistence, but there is controversy about its extent. There are questions about the quality of the data and the proper way to analyze it. We describe our data-analysis procedure in Appendix B. There are eleven strategy categories of hedge funds in TASS. (See, e.g., Boyson and Cooper (2004) or visit Credit Suisse/Tremont (www.hedgeindex.com) to find more about strategy categories.) We found positive persistence for all of them, using data from 2000 to 2005, but some estimated persistence factors were very low. We found five strategy categories of funds with significant persistence: (i) convertible, (ii) dedicated short bias, (iii) fixed income, (iv) fund of fund, and (v) others. Figure 2 is the scatter plot of two consecutive relative returns and the associated least-squares-fit with zero intercept for four of these fund categories. For these fund categories, the least-squares-fit slope, γ, ranges from 0.33 to 0.49, but there evidently is considerable randomness in the data. The persistence plays a big role in determining our Markov chain transition probabilities and, thus, our estimate of the lockup premium. We work hard to show how the lockup premium should depend on the persistence, and not on determining precisely what the persistence is. It is important to note, though, that the fund category is not the only element affecting persistence. Persistence can be measured for fund manager's tenure, asset size, fee structure, and so on, depending on the investor's judgement. As long as persistence is found or anticipated, our Markov chain models can be applied.

Having decided to use a Markov chain model, we must specify how the transitions take place over time. Given that hedge funds operate continually, it is natural to use continuous-time Markov chains (CTMC's), which allow transitions from one state to another in continuous time, even though we assume that reinvestment opportunities are restricted to being yearly. And that is what we do. However, discrete-time Markov chains (DTMC's) tend to be easier to work with, so we start with DTMC's with yearly transitions. With yearly transitions, we let state changes occur at the potential reinvestment times.

After we carry out the analysis for DTMC's, we show that it is also possible to carry out the corresponding analysis with CTMC's, but the analysis is more involved. We use nonlinear programming to efficiently solve the equations for model fitting. With a CTMC, state changes occur in continuous times, but reinvestment opportunities still occur yearly in discrete time. We thus use the transient one-year transition probabilities of the CTMC in an associated DTMC to describe what happens at the yearly reinvestment times. In addition to being more realistic, the CTMC model has the advantage that it can be fit to a wider range of hedge-fund

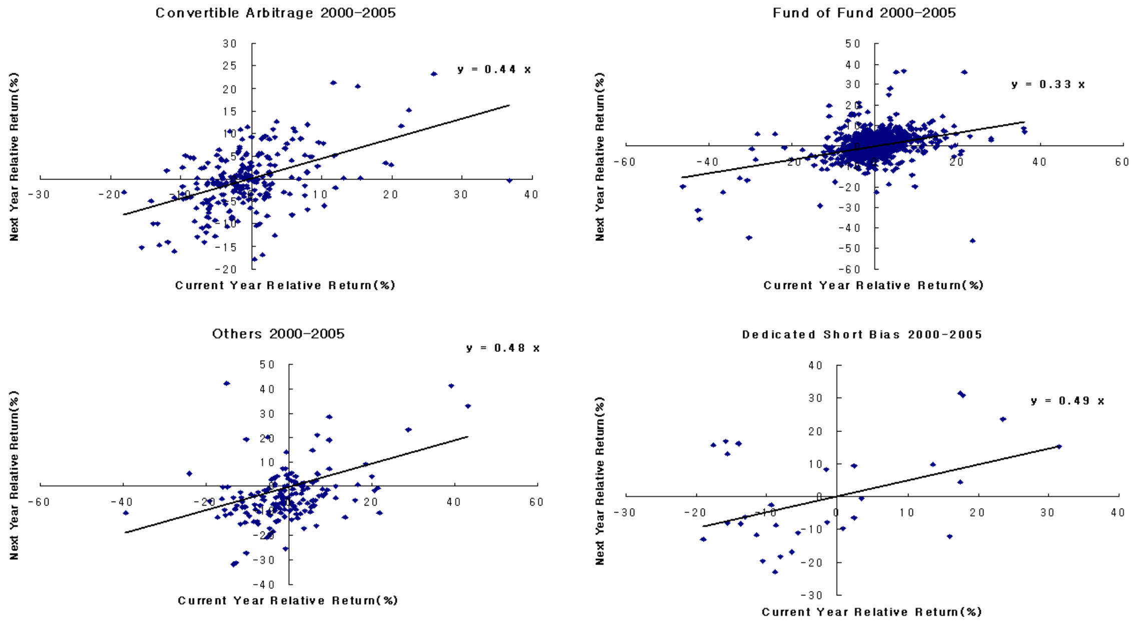


Figure 2: Scatter plots and associated least-squares lines for relative hedge fund returns in successive years from 2000 to 2005 for four strategy categories of hedge funds from the TASS database.

performance measures.

We show that the Markov chain models can be used to estimate how the lockup premium depends on the hedge-fund performance measures. Consistent with intuition, we show that the lockup premium increases with both the variance of the return and the persistence of the return, provided that the persistence is not too high. (There necessarily is no lockup premium with total persistence, when $\gamma = 1$.) What is less obvious, but consistent with intuition upon reflection, is that the lockup premium decreases with the hedge fund death rate. Of course the models do more: The models quantify the effect of these observable hedge fund performance measures on the lockup premium. For example, we show that the three-year lockup premium in the DTMC model is quite well approximated by the function $\psi(\delta, \gamma, \sigma) = 0.15\, \delta^{-0.11} \gamma^{0.74} \sigma^{1.00}$, where δ is the death rate, γ is the persistence and σ is the standard deviation of the yearly relative returns (under parametric assumptions to be explained later).

In §4 we present a simple analysis of the lockup premium based on persistence alone, without any Markov chains, which is appropriate when no hedge funds die. The more elaborate analysis in this paper is thus primarily intended to determine the effect of the death rate δ. Just as for the persistence, there is controversy about what is the actual death rate of hedge funds, with estimates ranging from $3-12\%$. Higher estimates follow from estimates of the median life of a hedge fund, as we explain in §4.1. That leads us to conclude that the death rate might be as high as $\delta = 0.09$. Just as with the persistence, we work hard to show how the lockup premium should depend on the death rate, and not on determining precisely what the death rate is. From our CTMC model, we conclude that a death rate of $\delta = 0.09$ per year makes the lockup premium about *half* of what it would be for $\delta = 0$. Figure 3 shows the lockup premium function for four values of δ. We conclude that the death rate is potentially a significant factor.

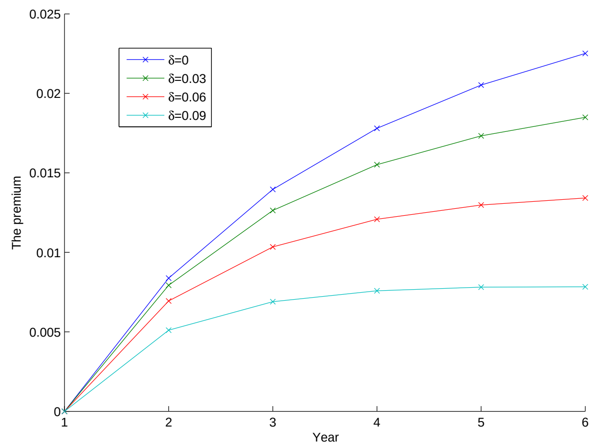


Figure 3: The lockup premium as a function of the extended lockup period, n, based on the CTMC model for four values of δ: $0.00, 0.03, 0.06$ and 0.09. The parameter values are in Table 2.

Organization of the paper. We start in §2 by reviewing the related literature on liquidity, including premiums for hedge fund lockup. In §3 we discuss persistence of hedge fund returns, reviewing the literature and analyzing data from the TASS database. In §4 we present the simple analysis of lockup premium based on persistence alone, without any Markov chains, based on no dying funds. This simple analysis provides a useful reference case, because it yields a simple formula. In §4.1 we discuss hedge-fund persistence and death rate in more detail. In §5 we introduce and analyze our three-state DTMC model. In §6 we show how the model parameters and the lockup premium depend on basic hedge fund performance measures. In §7 we consider the corresponding CTMC model. Finally, in §8 we draw conclusions. We present additional material in an appendix.

2. Liquidity Literature Review

There is a substantial literature on liquidity, including hedge fund lockup, but it mostly has a different character.

Liquidity premiums in asset pricing. The liquidity premium is well recognized as an important factor in asset pricing, but it is commonly measured by transaction cost; e.g, see Amihud and Mendelson (1986), Pastor and Stambaugh (2003), Chordia et al. (2001), and Eleswarapu and Reinganum (1993). For example, in the stock market, bid-ask spread is one measure of the liquidity premium. Amihud and Mendelson (1986) showed that there exists an increasing and concave relationship between the asset return and the bid-ask spread. Darar et al. (1998) confirmed this result, using the reciprocal of the stock turnover rate to measure the liquidity premium. More recently, Vayanos (2004) considered liquidity in an equilibrium model. He considered the liquidity premium in asset pricing with different transaction costs. He showed that as assets become more volatile, the required excess return from a riskless asset increases with the transaction costs.

Studies of liquidity have also been performed for the bond market; e.g., Amihud and Mendelson (1991), Warga (1992), Krishnamurthy (2002) and Longstaff (2004). For bonds, it is argued that there should exist a clear premium for liquidity, separate from the credit risk premium. Most-recently-issued U.S. Treasury bonds are considered the most liquid bonds available, among all bonds with similar conditions. Since US Treasury bonds are assumed to be riskless, they provide a natural way to measure the liquidity premium, without having to consider credit risk. The papers above study the liquidity premium by comparing the price of most-recently-issued US Treasury bonds (on the run) to the price of the bonds issued three months previously (off the run).

There are a few papers that are more closely related to what we do here, namely, Longstaff (1995, 2001) and Browne et al. (2003). These papers also view the liquidity premium as arising from the investor's inability to rebalance his portfolio in a timely way. Specifically, they define the liquidity premium as the additional required fixed return to compensate for the loss of the investor's utility from the inability to rebalance the investor's portfolio. They calculate the required liquidity premium as a function of the degree of risk averseness in the utility function, the market growth rate, and the liquidity restriction period. They rely heavily on mathematical models and mathematical analysis for this purpose. Unlike these references, we do not use utility functions. Our use of expected value corresponds to a linear utility function, which may be roughly appropriate for a fund of funds, which is a typical investor in hedge funds.

We conclude this section by mentioning Hayes (2006), which used a Markov chain model for a difference purpose – to develop a model for the maximum drawdown of hedge funds.

Empirical studies on hedge fund lockup. There is a growing literature on hedge funds, e.g., see Agarwal and Naik (2005), but only a few researchers have focused on hedge fund lockup. Liang (1999) found that the average hedge fund returns are related positively to the lockup periods from the analysis of Hedge Fund Research, Inc. (HFR) database. Aragon (2007) quantified the lockup premium for hedge funds empirically. He compared the hedge fund performance with and without extended lockup conditions. He estimated that the average difference in the annual returns is around $4 - 7\%$.

There also are empirical studies on the liquidity premium for funds other than hedge funds. For example, Ippolito (1989) conducted a similar study for mutual funds. There is a load-type mutual fund, which assesses sales charges. Ippolito (1989) found that the load-type mutual funds make approximately 3.5% higher return than no-load mutual funds.

In summary, from our investigation of the literature, we find that only a few papers - Longstaff (1995, 2001) and Browne et al. (2003) - have interpreted liquidity premium as quantification of the cost of a restricted rebalance opportunity. We found no previous papers employing models calibrated to data in order to estimate the liquidity premium. And none of the papers have used Markov chains, with the exception of Derman (2006), which is a preliminary account of the research reported here.

3. Persistence of Hedge Fund Returns

We specify how hedge funds perform by looking at the *relative return* of a fund, which is the difference between the annual return of the fund and the average annual return of all hedge funds in that hedge fund strategy category, where return is measured as a percentage. We do that to factor out the performance of the market as a whole. We evaluate hedge funds the way we might evaluate hedge fund managers, trying to identify whether or not their funds perform better than average. We estimate the persistence by doing a regression analysis on the hedge-fund-return data over several years from the TASS database.

The persistence literature. Before discussing our own analysis of data, we discuss the literature on performance persistence. Researchers have tried to take advantage of the two main hedge fund databases - TASS and HFR. In doing so, researchers have discovered that it is difficult to make unbiased estimates because reporting is voluntary, and some funds stop reporting, especially those performing poorly; e.g., see Jagannathan et al. (2006).

Despite the difficulty with biases in the hedge fund data, researchers have conducted studies. Although some researchers did not find evidence of performance persistence, others did. Brown et al. (1999) used a simple two-state categorization - win or lose - to measure performance persistence, recording a win if the fund beats the median return, but they did not find evidence of persistence. Boyson and Cooper (2004) carried out a similar analysis and still did not find evidence of persistence.

However, several papers found performance persistence for shorter periods ranging from a quarter to three years. Koh et al. (2003) used the method of Brown et al. (1999) for Asian hedge funds and found strong persistence in short horizons from monthly to quarterly. Agarwal and Naik (2000) and Jagannathan et al. (2006) used linear regression, like we do, as well as the previous two-way classifications. Agarwal and Naik (2000) did not provide regression slope explicitly but showed that depending on the strategy category of hedge fund, the percentage of funds which have statistically significant positive slope in regression ranges from 5 to 34 percent, where most of the strategy categories have around 20 percent. Using the same parametric linear regression and non-parametric two-way classifications, Agarwal and Naik (2000) claimed that the evidence of persistence is strongest for the shorter quarterly time periods. On the other hand, Edwards and Caglayan (2001) found strong persistence in over 1-2 years from the Managed Account Reports (MAR/Hedge) data. Furthermore, Jagannathan et al. (2006) found a significantly high performance persistence for a three-year period in his empirical study with HFR data. Jagannathan et al. (2006) carefully took account of the bias from voluntary reports and did regression analysis for the relative returns for three consecutive years. Using generalized method-of-moment (GMM) estimation, they found a statistically significant persistence factor of 0.56 for three-year period.

There also exists indirect evidence of performance persistence from the study of hedge-fund liquidation or survival. Brown et al. (2001) indirectly supported performance persistence when they found that a negative aggregated return over the previous two years increases the probability that a fund will liquidate. Furthermore, ter Horst et al. (2001) concluded that hedge-fund survival is strongly related to historical performance. Baquero et al. (2005)

conducted probit regression analysis of hedge-fund liquidation. They found that funds with high returns are much less likely to liquidate than funds with low returns from quarterly return data, which again indirectly supports persistence. We lastly remark that from the study of bid-ask spread in the stock market, Roll (1984) claimed that bid-ask spread increases as the price change is more serially correlated. As mentioned in §2, bid-ask spread is one of the representative measures of liquidity.

Our regression analysis. We conduct linear autoregression analysis with the TASS hedge fund performance data to find the best linear regression line between two consecutive year's relative returns. Specifically, letting the current year's (annual) relative return be denoted by R_c and the next year's relative return be denoted by R_n, we find the slope γ for the line $R_n = \gamma \cdot R_c$, which produces the minimum sum of squared errors.

The actual data analysis procedure is somewhat complicated; the key features are described in Appendix B. There are eleven strategy categories of hedge funds in the TASS data. We found positive persistence for all of these, but some estimated persistence factors were very low. (We think that better data should show higher persistence.)

We found five strategy categories of funds with significant persistence: (i) convertible, (ii) dedicated short bias, (iii) fixed income, (iv) fund of fund, and (v) others. Figure 2 is the scatter plot of two consecutive relative returns and the associated least-squares-fit with zero intercept for four of these fund categories. For these fund categories, the least-squares-fit slope, γ, ranges from 0.328 to 0.488.

A different way to estimate the persistence factor is to look at the ratio of the next-year average returns to the current-year average return, restricting attention to the returns that are positive in the current year. You get the same estimate when you repeat that procedure, but instead restrict attention to the returns that are negative in the current year. See Appendix C for the details.

4. Estimating the Lockup Premium from Persistence Alone

In this section we show how persistence alone, without any Markov chains, can be used to generate an estimate of the lockup premium. This simple analysis depends on four assumptions:

1. There is a single persistence factor γ.

2. We can ignore the phenomenon of hedge funds dying.

3. The returns each year are normally distributed with a fixed variance σ^2.

4. The performance of a fund is considered good if its annual return exceeds the average annual return.

The first two assumptions imply that the expected relative returns over time evolve *linearly*, enabling us to derive a simple *no-death lockup premium* as a function of the expected excess return of a good fund. The last two assumptions enable us to determine the expected excess return of a good fund. The third assumption can be weakened, but some analogous assumption is needed. The fourth assumption is just one possible case; it can easily be varied without altering the rest of the analysis.

The no-death lockup premium. We assume that the hedge fund starts off in a good state having just earned a relative return $Y_G > 0$, to be specified below. Let R_n be the expected relative return in the n^{th} year. The assumed γ persistence implies that the expected relative return at the end of the first year is $R_1 = \gamma Y_G$, for $0 \leq \gamma < 1$. The notion of γ persistence, with no funds dying, implies that we can *recursively* determine the expected relative return in successive years by

$$R_n = \gamma \cdot R_{n-1} = \gamma^n \cdot Y_G, \quad n \geq 1 . \tag{4.1}$$

As a consequence of (4.1), the total expected relative return up to the n^{th} year is

$$\sum_{i=1}^{n} R_i = \frac{\gamma Y_G (1-\gamma^n)}{1-\gamma} \ , \ 0 \leq \gamma < 1 . \tag{4.2}$$

Based on this simple analysis, we can compare the expected relative return from an n-year lockup with the expected relative return from a 1-year lockup in order to calculate the lockup premium. Under a 1-year lockup, investors have a chance to replace all sick funds with good funds at the end of each year. If they do, the expected return each year is the same as in the first year: $R_1 = \gamma Y_G$. Thus, at the end of the n^{th} year, the total expected relative return is simply $n\gamma Y_G$. On the other hand, under an n-year lockup, the fund just evolves without replacement up to the n^{th} year, as in (4.2). We assume that after the n^{th} year, the funds with 1-year and n-year lockups are both replaced by funds with the same 1-year lockup, so that there necessarily will be no difference in a fund's return after the n^{th} year.

Letting C_n be the total cumulative difference in expected return up through year n, we thus have

$$C_n = \sum_{i=1}^{n} (R_1 - R_i) = nR_1 - \sum_{i=1}^{n} R_i = \gamma Y_G \left(n - \frac{1-\gamma^n}{1-\gamma} \right) \ . \tag{4.3}$$

The lockup premium, denoted by $A_n \equiv A_n(\gamma)$, is the average annual difference. By (4.3), the *no-death lockup premium formula* is

$$A_n \equiv \frac{C_n}{n} = \gamma Y_G \left(1 - \frac{1-\gamma^n}{n \cdot (1-\gamma)}\right), \quad n \geq 1 \ , \tag{4.4}$$

which is a concave increasing function in n for each γ, $0 < \gamma < 1$, and a concave function of γ for each $n \geq 1$. The lockup premium $A_n(\gamma)$ is *not* an increasing function of γ overall; e.g., for $n = 2$, $A_n(\gamma) = Y_G\gamma(1-\gamma)/2$, which is increasing for $0 < \gamma < 1/2$, but decreasing for $1/2 < \gamma < 1$. More generally, $A_n(\gamma) = 0$ for both $\gamma = 0$ and $\gamma = 1$, with $A_n(\gamma) > 0$ for $0 < \gamma < 1$. However, the lockup premium function $A_n(\gamma)$ is increasing in γ for all sufficiently small γ, for each $n \geq 1$.

From (4.4), we see that $A_1 = 0$, $A_n \to \gamma Y_G$ as $n \to \infty$, and we have the bounds

$$\gamma Y_G \left(1 - \frac{1}{n(1-\gamma)}\right) \leq A_n \leq \gamma Y_G \left(1 - \frac{1}{n}\right), \quad n \geq 1 \ , \tag{4.5}$$

which yield convenient approximations. For large n or small γ, the lower bound is an accurate approximation.

The excess return from a good fund. The no-death lockup premium function $A_n(\gamma)$ clearly shows how the lockup premium depends on the three quantities: the length n of the extended lockup period, the persistence factor γ and the expected excess return of a good fund, Y_G. Clearly, n is directly observable, and we have seen how we can estimate γ, but it remains to specify Y_G.

However, if we define Y_G as the expected excess return of a good fund and apply the last two assumptions, then we can calculate Y_G. Letting $N(m, \sigma^2)$ denote a normally distributed random variable with mean m and variance σ^2, we have

$$Y_G = E[N(0,\sigma^2)|N(0,\sigma^2) > 0] = E[|N(0,\sigma^2)|] = \sigma E[|N(0,1)|] = \sqrt{2/\pi}\sigma \approx 0.8\sigma \ . \tag{4.6}$$

We can combine (4.4) and (4.6) to obtain the following general no-death lockup premium function

$$A_n(\gamma, \sigma) = 0.8\sigma\gamma \left(1 - \frac{1-\gamma^n}{n \cdot (1-\gamma)}\right), \quad n \geq 1 \ . \tag{4.7}$$

With assumptions 3 and 4, we see that the no-death lockup premium should be directly proportional to the standard deviation σ. Assumption 4 plays a key role in getting the simple formula (4.6), but we can generalize for arbitrary boundary point U, using the following formula for the conditional expectation of a normal random variable:

$$E[N(m,\sigma^2)|a \leq N(m,\sigma^2) \leq b] = m + \sigma \frac{[\phi\left((a-m)/\sigma\right) - \phi\left((b-m)/\sigma\right)]}{[\Phi\left((b-m)/\sigma\right) - \Phi\left((a-m)/\sigma\right)]} \tag{4.8}$$

for $-\infty \leq a < b \leq +\infty$, based on the relation $x\phi(x) = -\phi'(x)$ where ϕ is the standard normal density; e.g., see Prop. 18.3 of Browne and Whitt (1995). From formula (4.8), we see that Y_G will *not* be proportional to σ if we change the upper boundary point U.

We emphasize that, even under assumption 4 above, having A_n be directly proportional to σ depends critically on the third ceteris-paribus assumption made above. Since we are free to choose the monetary units, we can choose to define all returns relative to the standard deviation σ, which must be in the same units as the returns. In that sense, the lockup premium is *automatically* proportional to σ. The proportionality conclusion becomes more meaningful when we assume that the distribution of returns depends on σ as a simple scale factor, as provided by assumption 3 above. We need to impose a strong condition on the way the return distribution changes with σ in order to deduce the desired proportionality conclusion. The normality is only used to compute the precise value of the mean.

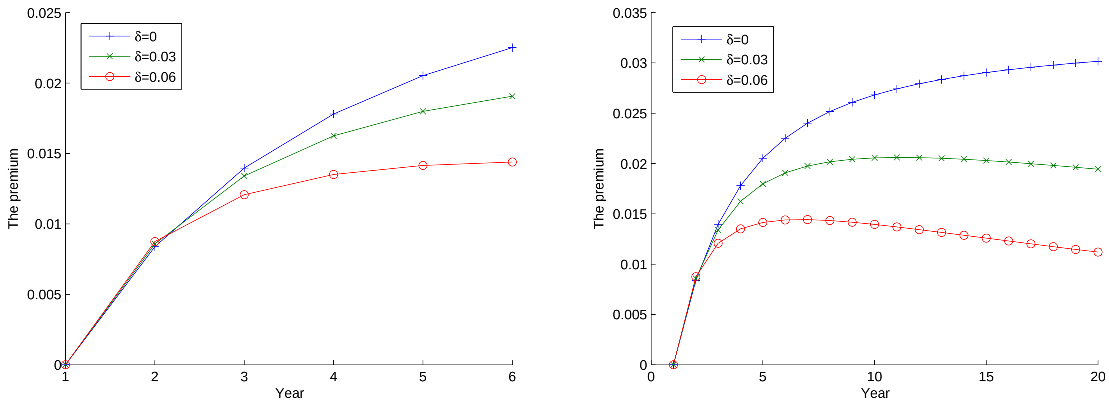
Relating to the calibration by Markov chains. We remark that the Markov chain model calibration will also produce its own estimates of the excess return Y_G, but we will find that analysis yields similar conclusions. Indeed, our main numerical example has $Y_G = 0.67\sigma$. We remark that we can obtain exactly that value if we take Y_G to be the *median* of the positive relative returns, because the median of the random variable $|N(0,1)|$ is 0.67.

Anticipating our future numerical examples with Markov chains, we plot our estimate for the lockup premium in Figure 4 for the case $\gamma = 0.5$, $\sigma = 0.1$ and $Y_G = 0.067$. Our estimate without death appears as the upper curve in Figure 4. We see that the lockup premium increases toward the limit $Y_G/2 = 0.0335$ as n increases.

We also plot two curves for positive death rates δ, obtained using the DTMC model in §5. The plotted cases for $\delta = 0.03$ and $\delta = 0.06$ show the importance of going beyond the no-death model. Consistent with Figure 4, we will see that the lockup premium is decreasing in the hedge fund death rate with our Markov chain model. Consequently, formulas (4.4) and (4.7) in this section, derived under the assumption of zero death rate, provide upper bounds on our estimated lockup premium with positive δ.

4.1. Important Hedge-Fund Performance Measures

Our Markov chain model will depend critically on the persistence of returns and the hedge-fund death rate. So we discuss these performance measures further now.



(i) From 1 to 6 years (ii) From 1 to 20 years

Figure 4: The lockup premium function for the DTMC model for three values of the hedge-fund death rate δ. The remaining model parameters are $Y_G = 0.067$, $Y_S = -0.15$, $Y_D = -0.20$ and $\gamma_G = \gamma_S = \gamma = 0.5$.

Two persistence factors: γ_G and γ_S. In equations (5.5) and (5.6) below we will introduce two state-dependent persistent factors γ_G and γ_S, instead of just the single γ, as we had in §4. Clearly, this generalization is important if the persistence factors for the two states do in fact differ significantly. To illustrate what actually may happen, Figure 5 shows the results of an regression analysis applied two consecutive-year relative returns for positive and negative parts of the current relative-return data separately. Figure 5 shows a significant difference in the slope of regression line for several fund categories, suggesting that it may be important to use separate state-dependent persistence factors.

The stationary death rate δ. We calibrate our models by specifying the overall annual death rate, denoted by δ. Unfortunately, estimating the death rate from the TASS database is difficult, in part because poorly performing funds often stop reporting, but funds also stop reporting for other reasons, e.g., because they seek no new investors.

After checking the reasons for funds being terminated in the HFR data, Rouah (2006) concluded that, after removing these biases, $3-5\%$ of the hedge funds leave the database each year because of failure. As noted in §3, Park (2006) estimated that the fund death rate is only 3.1 %, even though the total attrition rate from the TASS database was 8.7 % , based on her analysis from 1995 to 2004.

The death rate is closely related to the survival probability and median life of the fund. Clearly, as the death rate increases, the survival probability and the median life decrease. Since median life is more easily observable, it is convenient to verify the death rate of our model

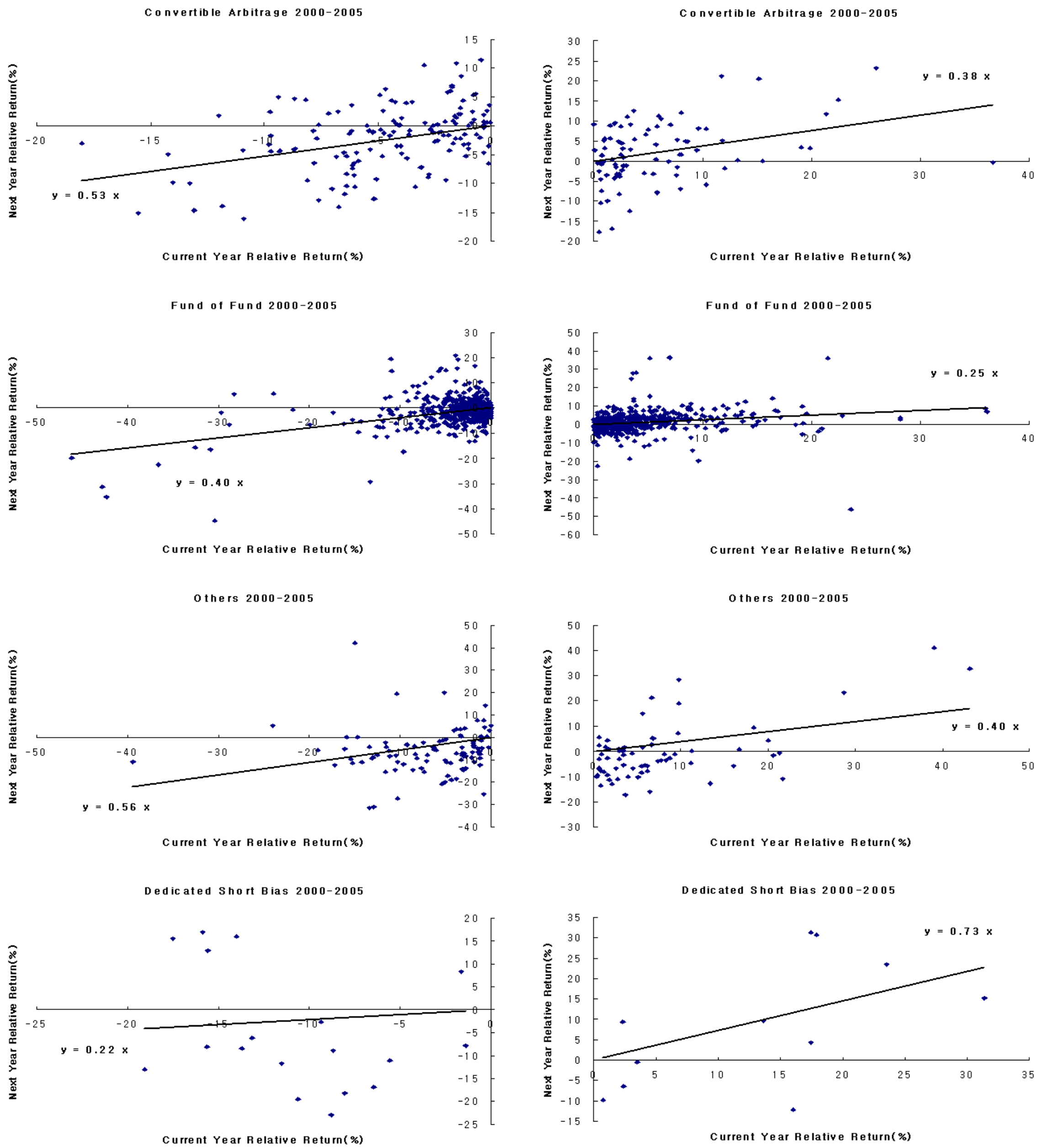


Figure 5: Scatter plots and least-squares lines for positive current relative returns and negative current relative returns of hedge funds from 2000 to 2005 in four categories: (i) convertible bond, (ii) fund of fund, (iii) others, and (iv) dedicated short bias.

through the median life in the hedge fund data.

One way to check the validity of the model is to calculate the survival probability curve produced by the model. In terms of the transition matrix P to be introduced in (5.1). the *probability of surviving* n *years* is $S_n = P^n_{G,G} + P^n_{G,S}$ for $n \geq 1$.

Figure 6 shows the survival probability curve for the DTMC model when $\delta = 0.03$ and 0.06. When $\delta = 0.03$, about 90% survive for 5 years, whereas the survival probability goes down to around 80% when $\delta = 0.06$. If we increase δ above 0.07, then we are unable to fit the DTMC model.

Studies estimating the median survival time of hedge funds were discussed in §3. In addition, Gregoriou (2002) estimated that median survival time of all hedge funds is 5.5 years, depending on factors such as millions managed, performance fee, leverage, minimum purchase and also on the redemption period. More recently, Rouah (2006) reported estimates of median survival time due before liquidation as ranging from 5.8 to 7.4 years based on the HFR data and from 7.2 to 17.4 years based on the TASS database. This last observation by Rouah (2006) suggests that our model with $\delta = 0.06$ may reasonably approximate the fund's performance.

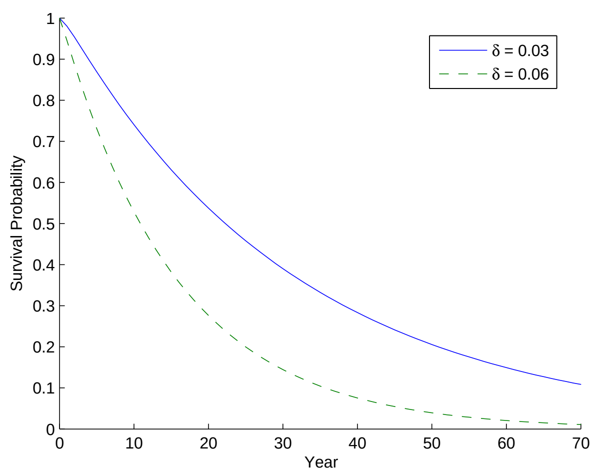


Figure 6: The survival probability for the DTMC model when $\delta = 0.03$ and 0.06, for parameter values given in the Table 1.

5. The Discrete-Time-Markov-Chain Model

We start in §5.1 by defining the basic three-state DTMC model, which has six parameters. Next in §5.2 we introduce four equations that the six parameters must satisfy, based on standard hedge fund performance measures. In §5.3 we develop explicit formulas for the three parameters appearing in the DTMC transition probabilities. In §5.4 we show how to calculate all the parameters after specifying two of the relative returns. We present numerical examples in §5.5. Finally, we show how to calculate the lockup premium in §5.6. Paralleling our treatment in §4, in §D in the Appendix we introduce a related two-state DTMC based on the assumption of zero death rate. That simplification is appealing because the formulas are more elementary.

5.1. The Basic DTMC Model

As indicated in the introduction, we let our Markov chain models have three states: good, sick and dead. We model the changing fund state over time as a DTMC, as in Chapter 4 of Ross (2003). We let time be discrete, with the unit of time representing one year. The initial DTMC is an *absorbing Markov chain*, with the D state being the sole absorbing state; once a fund becomes dead, it remains dead forever. We consider a transition matrix depending on three parameters: p, q and r:

$$P = \begin{array}{c} G \\ S \\ D \end{array} \begin{pmatrix} p & 1-p & 0 \\ q & r & 1-q-r \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

which corresponds to the following diagram: We have assumed that it is impossible to transition

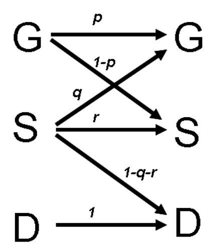


Figure 7: Transition probabilities in the absorbing Markov chain

from good to dead in a single year, thus eliminating one parameter.

We now move on to consider an associated *ergodic Markov chain*, having a non-degenerate limiting steady-state distribution, by assuming that a new hedge fund appears in the good state to replace a dead hedge fund right after it dies. This can be done with the new three-state DTMC transition matrix

$$P = \begin{array}{c} G \\ S \\ D \end{array} \begin{pmatrix} p & 1-p & 0 \\ q & r & 1-q-r \\ p & 1-p & 0 \end{pmatrix}. \tag{5.2}$$

In (5.2), the transition probabilities from a dead state are the same as from a good state, because a dead fund is immediately replaced by a good fund.

From the basic theory of DTMC's, as in Theorem 4.1 of Ross (2003), we obtain the steady-state probability vector $\pi \equiv (\pi_G, \pi_S, \pi_D)$ by solving $\pi = \pi P$ under the condition that $\pi_G + \pi_S + \pi_D = 1$. The stationary probability vector π for the transition matrix P in (5.2) is

$$\pi_G = \frac{q + p(1-q-r)}{2-p-r}, \ \pi_S = \frac{1-p}{2-p-r}, \ \pi_D = \frac{(1-p)(1-q-r)}{2-p-r}. \tag{5.3}$$

Our DTMC model uses both transition matrices. We use the absorbing transition matrix in (5.1) when we compute the expected return of a fund, while we use the ergodic transition matrix in (5.2) when we calculate the steady-state death rate and performance variance.

We will act as if the fund earns a state-dependent fixed average relative return in each state. We must specify these relative returns. Let Y_G, Y_S and Y_D denote the expected relative returns in the states G, S and D, respectively. Overall, we have six parameters: p, q, r, Y_G, Y_S and Y_D.

5.2. The Four Model-Fitting Equations

We first consider the death rate, which is defined as the proportion of live funds (in a good or sick state) that die during one transition period, which we take to be one year. For the transition matrix in (5.1), only sick funds can die in one transition. Thus, the death rate equals the product of the steady-state probability that a fund is sick times the transition probability from sick to dead. By (5.1) and (5.3), the *death rate* is

$$\delta = \pi_S \cdot P_{S,D} = \frac{1-p}{2-p-r}(1-q-r) = \pi_D \ . \tag{5.4}$$

We now introduce two equations determined by the persistence. For greater model flexibility, we allow different persistence in states G and S. The two *DTMC-persistence* equations are:

$$\gamma_G \cdot Y_G = p \cdot Y_G + (1-p) \cdot Y_S \tag{5.5}$$

and

$$\gamma_S \cdot Y_S = q \cdot Y_G + r \cdot Y_S + (1-q-r) \cdot Y_D \ . \tag{5.6}$$

We explain these DTMC-persistence equations as follows: In equation (5.5), the fund starts with state G; in equation (5.6) the fund starts with state S. The left side describes expected return in the next period calculated using the relevant persistence factor, whereas the right side calculates expected return in the next period using the transition probabilities of the DTMC in (5.1).

Our fourth equation is for the steady-state variance of the annual returns. Since we are working with returns relative to the mean, the variance of the steady-state return coincides with the second moment. Thus, the *variance* equation is

$$\sigma^2 = \pi_G \cdot Y_G^2 + \pi_S \cdot Y_S^2 + \pi_D \cdot Y_D^2 \ . \tag{5.7}$$

5.3. Explicit Formulas for the Transition Probabilities

We now derive formulas for the DTMC transition probability parameters p, q and r in terms of Y_G, Y_S, Y_D, γ_G, γ_S and δ using the three equations (5.4), (5.5) and (5.6).

The three formulas. Assuming that γ_G, γ_S, δ, Y_G, Y_S and Y_D are specified, the three equations in (5.4), (5.5), and (5.6) produce three equations in the three unknowns p, q and r. We first observe that the variable p can be solved from the single equation in (5.5), because that is a single equation for the single unknown variable p. The solution is

$$p = \frac{\gamma_G \cdot Y_G - Y_S}{Y_G - Y_S} \ . \tag{5.8}$$

Having found the explicit expression for p in (5.8), we substitute in for p to obtain two equations in the remaining two unknowns q and r. Indeed, given p, we can rewrite each of the two remaining equations to express q directly as a function of r. First, from (5.4), we get

$$q \equiv q(r) = 1 - r - \frac{\delta(2-p-r)}{1-p} = 1 - \delta\left(\frac{2-p}{1-p}\right) - r\frac{(1-p-\delta)}{(1-p)} \ . \tag{5.9}$$

Since $\delta < 1 - p$ by (5.4), the function $q(r)$ in (5.9) is necessarily strictly decreasing in r.

Next, (5.6) can be rewritten as

$$q \equiv q(r) = \frac{\gamma_S \cdot Y_S - Y_D - r(Y_S - Y_D)}{Y_G - Y_D} = \frac{(\gamma_S - r)Y_S - (1-r)Y_D}{Y_G - Y_D} \ . \tag{5.10}$$

Combining the two equations (5.9) and (5.10), we get an explicit expression for r, first in terms of p and then in terms of the basic model parameters, namely,

$$r = \frac{\left(\frac{1-p-\delta(2-p)}{1-p}\right) - \left(\frac{\gamma_S \cdot Y_S - Y_D}{Y_G - Y_D}\right)}{\left(\frac{1-p-\delta}{1-p}\right) - \left(\frac{Y_S - Y_D}{Y_G - Y_D}\right)} = \frac{\left(\frac{(1-\delta)(1-\gamma_G)Y_G - \delta(Y_G - Y_S)}{(1-\gamma_G)Y_G}\right) - \left(\frac{\gamma_S \cdot Y_S - Y_D}{Y_G - Y_D}\right)}{\left(\frac{(1-\gamma_G)Y_G - \delta(Y_G - Y_S)}{(1-\gamma_G)Y_G}\right) - \left(\frac{Y_S - Y_D}{Y_G - Y_D}\right)} \tag{5.11}$$

To be feasible, we of course need $0 \leq q \leq 1 - r$ and $0 \leq r \leq 1$. Formulas (5.9) and (5.11) simplify when $\delta = 0$; see §D in the Appendix.

Commentary. We now want to determine what parameter values can occur. Figure 8 shows the three parameters as a function of δ with $Y_G = 0.067$, $Y_S = -0.15$, $Y_D = -0.20$ and $\gamma_G = \gamma_S = 0.5$.

From (5.8) we see that p is a linear function of γ_G with positive slope $Y_G/(Y_G - Y_S)$. If $Y_S \leq 0$, then we necessarily have $\gamma_G < p < 1$. The minimum possible value of p, attained when $\gamma_G = 0$, is $|Y_S|/(Y_G + |Y_S|)$. For example, if $Y_G = 0.05 > 0 > Y_S = -0.15$, then the minimum

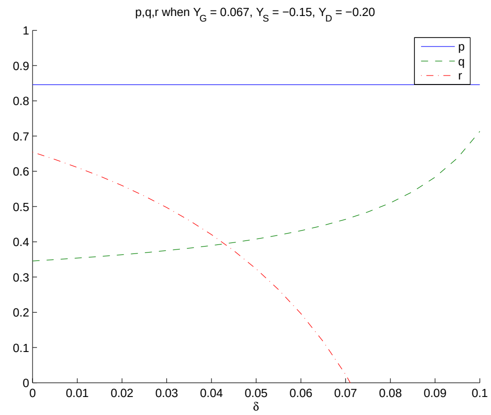


Figure 8: The DTMC parameter values p, q and r as a function of δ when $Y_G = 0.067$, $Y_S = -0.15$, $Y_D = -0.20$ and $\gamma_G = \gamma_S = 0.5$

value of p is 0.75 (at $\gamma_G = 0$) and the slope is 0.25. On the other hand, if $Y_G > Y_S > 0$, then we must have $p \leq \gamma_G$. If, instead, $Y_G > Y_S > 0$, then we require that $\gamma_G \cdot Y_G > Y_S$.

From (5.8), we see that p is independent of δ. Since (5.4) implies that $\delta < 1 - p$, there is an upper bound on the possible δ, consistent with Figure 8. Moreover, that inequality can be restated as $p < 1 - \delta$. When combined with (5.8), that yields an upper bound on γ_G, which is strictly less than 1: $\gamma_G \leq (1-\delta) + \delta(Y_S/Y_G)$. For $Y_G = 0.067$ and $Y_S = -0.150$, $\gamma_G \leq 1 - 3.23\delta$.

Under the general condition that $Y_G > Y_S > Y_D$, we see that $q \equiv q(r)$ via (5.10) is a strictly decreasing function of r. The largest possible value of q occurs for $r = 0$, which is $(\gamma_S \cdot Y_S - Y_D)/(Y_G - Y_D)$. In order for q to be feasible (nonnegative), we must have that largest possible value be nonnegative. Hence to have a feasible nonnegative value of q, we must have $\gamma_S \cdot Y_S \geq Y_D$. That is always satisfied provided that $Y_D \leq 0$ (given that $Y_G > Y_S > Y_D$).

From (5.10) alone, we can find an upper bound on r in terms of γ_S, Y_S and Y_D. If $0 > Y_S > Y_D$, then we must have $(1-r)|Y_D| \geq (r - \gamma_S)|Y_S|$, so that

$$r < \frac{|Y_D/Y_S| + \gamma_S}{|Y_D/Y_S| + 1} < 1 \quad \text{for} \quad 0 < \gamma_S < 1 \; , \tag{5.12}$$

where $|Y_D/Y_S| > 1$. On the other hand, if $Y_S \geq 0 > Y_D$, then we have

$$r < \frac{(|Y_D|/Y_S) - \gamma_S}{(|Y_D|/Y_S) + 1} < 1 \; , \tag{5.13}$$

where now $|Y_D|/Y_S$ can assume a wide range of values.

When $Y_G > 0 \geq Y_S > Y_D$, r has the form $r = (a - B)/(A - b)$, where $a < A$ and $b < B$, so that we always have $r < 1$. We then have $r > 0$ if and only if either $a > B$ or $A < b$; r is negative otherwise. To have $r > 0$, we must have

$$a - B \equiv \left(\frac{(1-\delta)(1-\gamma_G)Y_G - \delta(Y_G - Y_S)}{(1-\gamma_G)Y_G} \right) - \left(\frac{\gamma_S \cdot Y_S - Y_D}{Y_G - Y_D} \right) > 0 \quad (5.14)$$

or

$$b - A \equiv \left(\frac{Y_S - Y_D}{Y_G - Y_D} \right) - \left(\frac{(1-\gamma_G)Y_G - \delta(Y_G - Y_S)}{(1-\gamma_G)Y_G} \right) > 0 \ . \quad (5.15)$$

Examination of (5.11) shows that there can be difficulties in r as $\gamma_G \uparrow 1$, because the term $\delta(Y_G - Y_S)/(1-\gamma_G)Y_G$ appearing in the terms a and A diverges as $\gamma_G \uparrow 1$.

In summary, from this analysis, we see that there is an upper limit on how high the death rate δ and the persistence γ can be. For the other parameters we consider, the maximal possible death rate is $\delta = 0.07$. We will see that the CTMC model allows higher values of δ, up to $\delta = 0.13$ for these parameter values.

5.4. Determining All Model Parameters

We now put everything together to develop an algorithm for computing all the model parameters.

An iterative algorithm. There are several ways we may proceed. We choose to specify Y_S and Y_D in addition to δ, γ_G, γ_S and σ. (This decision is supported by the fact that the model parameters are less sensitive to Y_S and Y_D than to Y_G, as we will see in §6.) Specifying these two quantities determines all the parameters. We then calculate the model parameters iteratively. We do so by guessing Y_G, which enables us to directly calculate the DTMC parameters p, q and r, and then the steady-state probability vector π. Given π, we can then calculate σ from (5.7). We then iterate until the calculated σ agrees with the initially specified value of σ.

Although it is not entirely evident from the equations, because π depends on Y_G too, our calculations indicate that σ is an increasing function of Y_G, so it is easy to find the appropriate value of Y_G, e.g., by performing bisection search. A simple plot of σ versus Y_G verifies this property, and reveals the appropriate value of Y_G. We illustrate in Figure 9 below for the special case in which $Y_S = -0.15$, $Y_D = -0.20$, $\gamma_G = \gamma_S = 0.5$ and $\delta = 0.03$.

Denominating in terms of σ. For additional insight, it is helpful to express our returns in units of the standard deviation σ. We can divide through by σ^2 in (5.7) to obtain

$$1 = \pi_G \cdot (Y_G/\sigma)^2 + \pi_S \cdot (Y_S/\sigma)^2 + \pi_D \cdot (Y_D/\sigma)^2 \ . \quad (5.16)$$

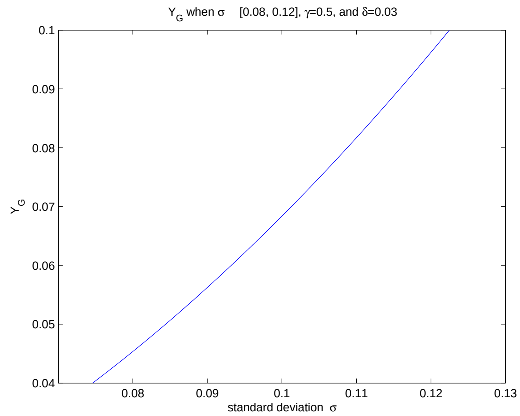


Figure 9: The standard deviation of relative return σ versus Y_G when $Y_S = -0.15, Y_D = -0.20$, $\gamma_G = \gamma_S = 0.5$, and $\delta = 0.03$.

Observe that the steady-state probability vector π in (5.3) and the death rate δ in (5.4) depend only on DTMC parameters p, q and r, while the equations (5.8), (5.10) and (5.11) for p, q and r are invariant under scale multiples of Y_G, Y_S and Y_D.

Paralleling Figure above, it is useful to see how Y_G/σ behaves as a function of σ when we fix Y_S/σ and Y_D/σ in addition to δ and γ. It turns out that, after fixing $Y_S/\sigma = -1.5$ and $Y_D/\sigma = -2.0$, the value of Y_G/σ is constant when $\delta = 0$ and almost constant (very weakly increasing) when $\delta > 0$. For the special case in which $Y_S/\sigma = -1.5, Y_D/\sigma = -2.0, \gamma_G = \gamma_S = 0.5$, and $\delta = 0.03$, $Y_G/\sigma \approx 0.685$ for σ ranging from 0.07 to 0.13. It is thus convenient and useful to set Y_S and Y_D proportional to σ. *We hereafter set $Y_S = -1.5\sigma$ and $Y_D = -2.0\sigma$ for our analysis.*

5.5. Numerical Examples

We now consider some numerical examples. Our base case is $\delta = 0.03, \gamma_G = \gamma_S = \gamma = 0.5, \sigma = 0.1$, $Y_S = -1.5\sigma = -0.15$, and $Y_D = -2.0\sigma = -0.20$. If we try $Y_G = 0.685\sigma = 0.0685$, then we get $p = 0.8432$, $q = 0.3719$, $r = 0.5030$, and $\sigma = 0.1001$.

Table 1 shows parameter values for various δ, γ_G, γ_S, with Y_S, Y_D and σ fixed as above, the return Y_G is calculated iteratively by the method above. The last line of the Table 1 shows that r is negative. Our numerical analysis shows that r reaches 0 and becomes negative when δ is above 0.07. The CTMC model is more flexible, providing a solution for $\delta \leq 0.13$.

Table 1: Parameter value sets

δ	γ_G	γ_S	σ	Y_G	Y_S	Y_D	Calculated σ	p	q	r
0.00	0.5	0.5	0.1	0.067	-0.15	-0.20	0.1002	0.8456	0.3456	0.6544
0.03	0.5	0.5	0.1	0.0685	-0.15	-0.20	0.1001	0.8432	0.3719	0.5030
0.06	0.5	0.5	0.1	0.070	-0.15	-0.20	0.1001	0.8409	0.4207	0.2282
0.07	0.5	0.5	0.1	0.075	-0.15	-0.20	0.1001	0.8401	0.4474	0.0796
0.00	0.6	0.4	0.1	0.076	-0.15	-0.20	0.1000	0.8655	0.3982	0.6018
0.03	0.6	0.4	0.1	0.077	-0.15	-0.20	0.1002	0.8643	0.4320	0.4069
0.06	0.6	0.4	0.1	0.0775	-0.15	-0.20	0.1000	0.8637	0.5068	-0.0127

5.6. The Lockup Premium Calculation

The lockup premium calculation is essentially the same as in §4, following (4.3) and (4.4), but using the DTMC in (5.2) to compute the expected relative returns for the n-year lockup. In particular,

$$A_n \equiv \frac{C_n}{n} \equiv R_1 - \frac{1}{n}\sum_{i=1}^{n} R_i = \gamma Y_G - \frac{1}{n}\sum_{i=1}^{n} \left(P^i_{G,G} \cdot Y_G + P^i_{G,S} \cdot Y_S + P^i_{G,D} \cdot Y_D\right) \;, \qquad (5.17)$$

where P^i is the i^{th} power of the transition matrix P in (5.2).

For example, if we set $\sigma = 0.1$, $Y_G = 0.685\sigma$, $Y_S = -1.5\sigma$, $Y_D = -2.0\sigma$, $\gamma_G = \gamma_S = \gamma = 0.5$ and $\delta = 0.03$, we get $p = 0.8432$, $q = 0.3719$ and $r = 0.5030$ from §5.5. The difference between a 2-year lockup and a 1-year lockup is 0.86 percentage points of return whereas the difference between a 3-year lockup and a 1-year lockup is 1.34 percentage points of return. Figure 4 in §4 shows the lockup premium function for three values of δ: 0.00, 0.03 and 0.06.

We remark that when $\delta = 0$, (5.17) reduces to (4.4). Using (5.7) with $\pi_D = 0$ for $\delta = 0$, we obtain $(Y_G/\sigma)(Y_S/\sigma) = 1$. Thus, if we fix $Y_S/\sigma = -1.5$, the lockup premium is directly proportional to σ, just as in §4; see Appendix D.

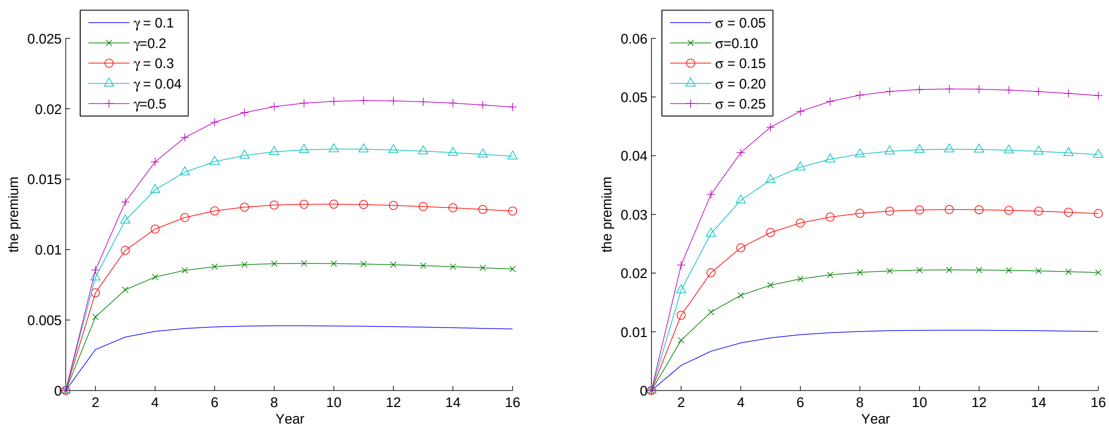
6. Sensitivity Analysis for the DTMC model

The mathematical models developed here are useful to estimate how the lockup premium depends on the different variables. We describe highlights of such analyses here and present more details in the appendix.

Our results here are related to the standard *base case* with $\gamma_G = \gamma_S = \gamma = 0.5$, $\sigma = 0.1$, $Y_S = -1.5\sigma$, $Y_D = -2.0\sigma$ and $\delta = 0.03$, as in the second row of Table 1.

Figure 10 (i) shows the lockup premium for five values of γ: 0.1, 0.2, 0.3, 0.4 and 0.5 while Figure 10 (ii) shows the lockup premium for five values of σ: 0.05, 0.10, 0.15, 0.20 and 0.25.

In both cases, these changes produce minor changes in Y_G and the other model parameters; see the Appendix.



(i) γ: 0.1, 0.2, 0.3, 0.4 and 0.5 (ii) σ: 0.05, 0.10, 0.15, 0.20 and 0.25

Figure 10: The lockup premium for the DTMC model in the base case with five values of γ and σ.

We next consider how the DTMC model parameters p, q and r depend on the other driving variables. To supplement Figure 8 and the commentary in §5.3, Figure 11 shows how these parameters p, q and r depend on γ (assuming $\gamma_G = \gamma_S = \gamma$) and each of the return values Y_G, Y_S and Y_D, taken one at a time. We see that the model becomes unstable if γ gets very large, but there is nice near-linear behavior for values of $\gamma \leq 0.5$. We also see that the parameters p, q and r are considerably more sensitive to Y_G than the other two returns Y_S and Y_D.

Lastly, we consider how the DTMC lockup premium for a fixed lockup period depends on three variables δ, γ, and σ. Figure 12 shows how the three-year lockup premium depends on two of the three variables while fixing the remaining variable. We see that the three-year lockup premium is reasonably well approximated by a linear function of γ and σ, respectively; there is concavity in γ but convexity in σ. Also, the three-year lockup premium is relatively insensitive to δ. However, we note that the CTMC model predicts greater impact of δ on the three-year lockup premium, as illustrated in Figure 3.

We remark that the lockup premium in the DTMC model can be approximated by a simple functional form of three variables δ, γ, and σ with the choice of $Y_S/\sigma = -1.5$ and $Y_D/\sigma = -2.0$. The approximation with a simple functional form is helpful to quickly estimate how the lockup premium changes if the variables change.

We had success fitting the simple product form of the three variables with an exponent for each variable for the fixed year lockup premium, denoted by $\psi^p(\delta, \gamma, \sigma) = a\ \delta^b \gamma^c \sigma^d$. After

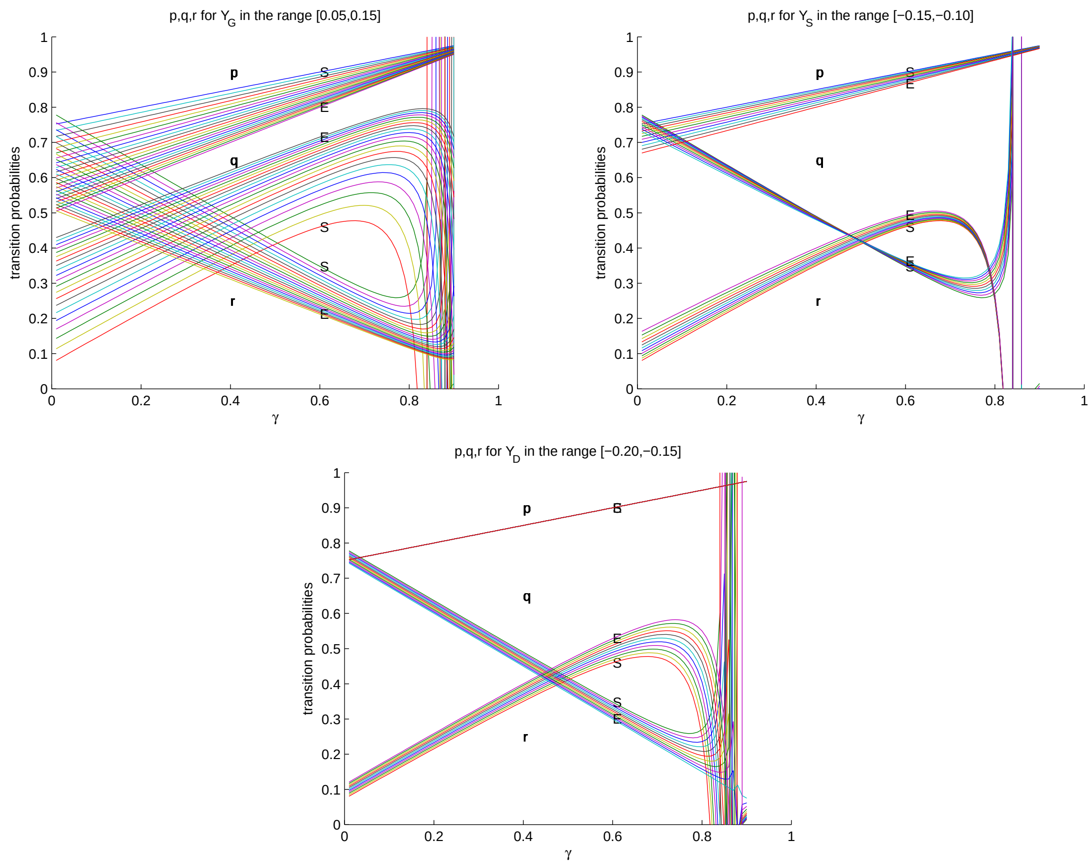
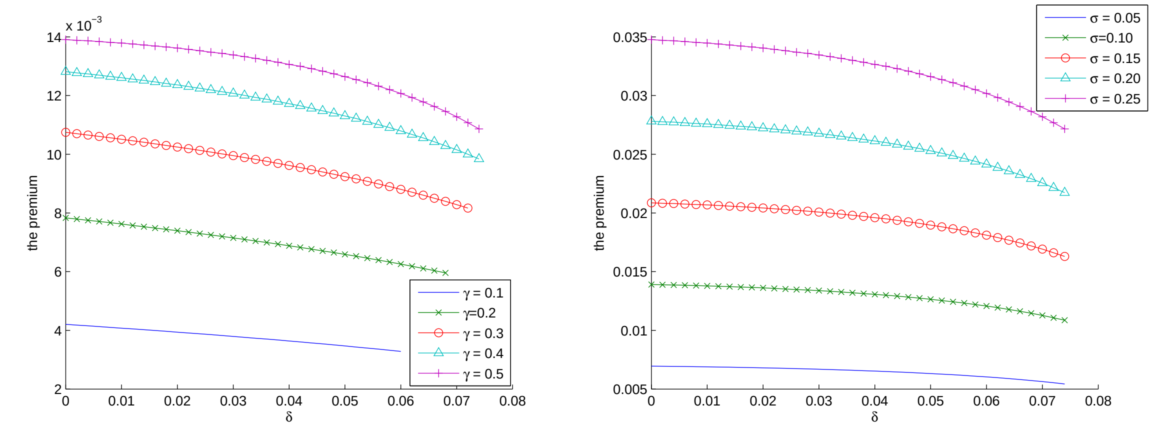


Figure 11: The parameters p, q and r as a function of γ in the base case for values of Y_G ranging from 0.05 (starting value, denoted by S) to 0.15 (ending value, denoted by E), Y_S ranging from -0.15 to -0.10, and Y_D ranging from -0.20 to -0.15

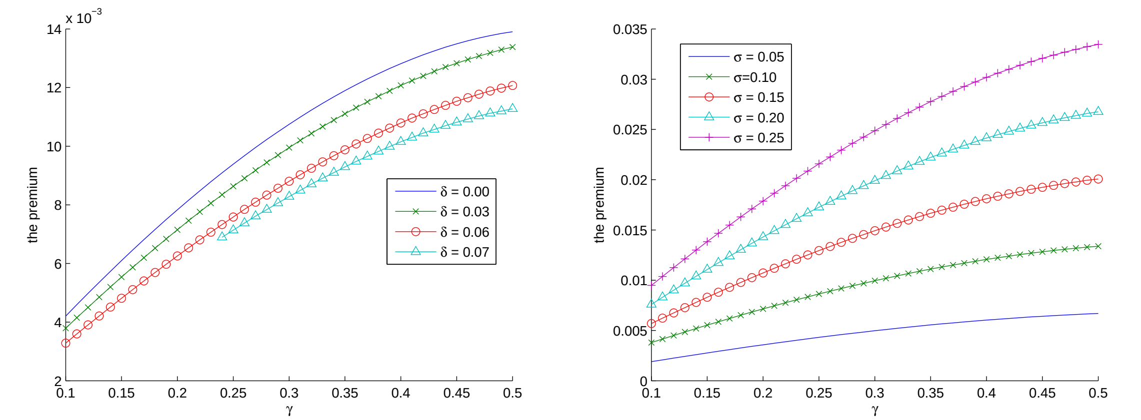
taking logarithms, we can easily apply linear regression for the lockup premium values in the DTMC model to estimate parameters a, b, c and d. By that method, the three-year lockup premium is approximated by $\psi^p_{(3)} = 0.15\ \delta^{0.11}\gamma^{0.74}\sigma^{1.00}$ with maximum error of 0.0039 in the base case $Y_S/\sigma = -1.5$ and $Y_D/\sigma = -2.0$. The product approximation can be extended to different lockup periods (n) and choice of Y_S/σ and Y_D/σ. See Appendix E.4 for further discussion.

7. The CTMC Model

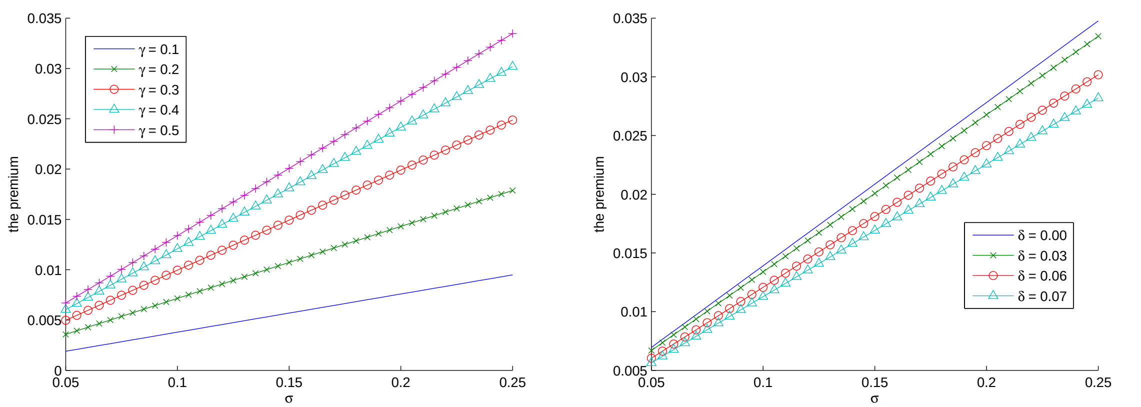
In the hope of providing better predictions, we now propose a more sophisticated model, which uses a CTMC - specifically a continuous-time birth-and-death (BD) process - and a DTMC; see Chapter 6 of Ross (2003) for background. Now the fund changes state in continuous time, but the investment updates take place in discrete time. We can study the model behavior as a function of the time T allowed for updates, if we wish, but we let $T = 1$ in our numerical



For $\delta = 0.00$ to 0.08, (i) $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$ with $\sigma = 0.1$ (ii) $\sigma = 0.05, 0.1, 0.15, 0.2, 0.25$ with $\gamma = 0.5$



For $\gamma = 0.1$ to 0.5, (iii) $\delta = 0.00, 0.03, 0.06, 0.07$ with $\sigma = 0.1$ (iv) $\sigma = 0.05, 0.1, 0.15, 0.2, 0.25$ with $\delta = 0.03$



For $\sigma = 0.05$ to 0.25, (v) $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$ with $\delta = 0.03$ (vi) $\delta = 0.00, 0.03, 0.06, 0.07$ with $\gamma = 0.5$

Figure 12: The three-year lockup premium for the DTMC model with $Y_S = -1.5\sigma, Y_D = -2.0\sigma$. The lockup premium does not exist if q or r becomes negative.

examples.

With the DTMC model, when a fund starts in a good state, at least two years are required for the fund to become dead. In contrast, with the CTMC model, a fund can become dead at any time. There is a cost, however: For the CTMC model, we are unable to fit the model parameters simply by solving three equations in three unknowns as we did for the DTMC in §5.3. Instead, we do the model fitting numerically. However, the parameter fitting for the CTMC model is not substantially harder than for the DTMC model, when we consider the iteration needed to find Y_G for given σ with the DTMC, discussed in §5.4.

In our proposed CTMC model we replace the three-state absorbing DTMC in (5.1) by a two-state absorbing BD process. The states now are G and S; we do not directly use the state D here, but we will be able to account for it. As usual, we specify the BD process by specifying its infinitesimal transition matrix Q. That means we specify the birth and death rates. Let μ_G be the death rate in G, the rate of transition down to state S from state G. Let λ_S be the birth rate in state S, the rate of transition up to state G from state S. Let μ_S be the death rate in state S, implicitly the rate of transition down to state D from state S. (The fund may leave state S to go to state D, but gets absorbed in D. We do not need to include the state D in our transition rate matrix.) Here is the infinitesimal transition matrix, with the parameters above:

$$Q = \begin{matrix} G \\ S \end{matrix} \begin{pmatrix} -\mu_G & \mu_G \\ \lambda_S & -(\lambda_S + \mu_S) \end{pmatrix} . \tag{7.1}$$

7.1. The Transition Matrix

We now want to derive the time-dependent transition probability matrix $P(t)$ for this BD process. It is well-known that $P(t)$ is the solution to the matrix ordinary differential equation

$$P(t)' = P(t)Q, \ P(0) = I \ , \tag{7.2}$$

where I is the identity matrix, so that $P(t)$ is the matrix exponential $P(t) = e^{tQ}$. If we diagonalize Q so that $Q = UDU^{-1}$, where D is a diagonal matrix and $UU^{-1} = I$, then we can write $P(t) = Ue^{tD}U^{-1}$; see §4.8 and the appendix of Karlin and Taylor (1975). Since D is a diagonal matrix, the i^{th} diagonal element of e^{tD} is related to the corresponding diagonal element of D, i.e., $(e^{tD})_{i,i} = e^{D_{i,i}t}$ for $t > 0$. Let $\Lambda(t)$ be a diagonal matrix of the form

$$\Lambda(t) = \begin{matrix} G \\ S \end{matrix} \begin{pmatrix} e^{\eta_G t} & 0 \\ 0 & e^{\eta_S t} \end{pmatrix} , \tag{7.3}$$

with the two parameters η_G and η_S being the eigenvalues of the matrix Q, while the columns of U are the associated right eigenvectors. The resulting formula for $P(t)$ is

$$P(t) = U\Lambda(t)U^{-1} \ . \tag{7.4}$$

The characterization (7.4) implies that $P_{i,j}(t) = A_{i,j}e^{\eta_1 t} + B_{i,j}e^{\eta_2 t}$ for $t \geq 0$ and all state pairs (i, j), where η_1 and η_2 are the eigenvalues of Q and $A_{i,j}$ and $B_{i,j}$ are appropriate constants.

Since $P(0) = I$, we necessarily have $A_{i,i} + B_{i,i} = 1$ for $i = 1, 2$ and $A_{i,j} + B_{i,j} = 0$ for $i \neq j$. If $0 > \eta_1 > \eta_2$, then asymptotically $P_{i,j}(t) \sim A_{i,j} e^{-\eta_1 t}$ as $t \to \infty$, which means that the ratio approaches 1. As a consequence, necessarily $A_{i,j} > 0$ for all state pairs (i, j); $B_{i,j} = -A_{i,j}$ for $i \neq j$.

As usual, we find the eigenvalues of Q by finding the determinant of $\eta I - Q$. The characteristic polynomial as a function of the variable η is the quadratic equation

$$(\eta + \lambda_S + \mu_S)(\eta + \mu_G) - \lambda_S \mu_G = 0 \ , \tag{7.5}$$

which has two strictly negative roots, as required for the formula in (7.3) to yield bonafide probabilities. In particular, solving the quadratic equation, we obtain

$$\eta = \frac{-(\lambda_S + \mu_S + \mu_G) \pm \sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S \mu_G}}{2} \ . \tag{7.6}$$

Since the term inside the square root can be rewritten as $(\mu_G - \mu_S)^2 + \lambda_S^2 + 2\mu_G \lambda_S + 2\lambda_S \mu_S$, it is nonnegative. The first term clearly dominates the square root in absolute value. So we indeed have two negative roots.

Now we find eigenvectors corresponding to the eigenvalues in (7.6). Given eigenvalues, the eigenvectors form the null space of $(Q - \eta I)$, i.e., a matrix U such that $(Q - \eta I)U = 0$. We arrange eigenvalues η_G, η_S as η matrix:

$$\eta = \begin{pmatrix} \eta_G \\ \eta_S \end{pmatrix} = \begin{pmatrix} \frac{-(\lambda_S+\mu_S+\mu_G)-\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}}{2\lambda_S} \\ \frac{-(\lambda_S+\mu_S+\mu_G)+\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}}{2\lambda_S} \end{pmatrix} \ . \tag{7.7}$$

Such an eigenvector matrix U, where the columns of U are eigenvectors of Q, can be found by algebraic manipulation or by a symbolic calculation package such as Mathematica. One such eigenvalue matrix is

$$U = \begin{pmatrix} \frac{-(\lambda_S+\mu_S+\mu_G)-\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}}{2\lambda_S} & \frac{-(\lambda_S+\mu_S+\mu_G)+\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}}{2\lambda_S} \\ 1 & 1 \end{pmatrix} \ . \tag{7.8}$$

Its inverse matrix is then

$$U^{-1} = \begin{pmatrix} -\frac{\lambda_S}{\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}} & \frac{\lambda_S-\mu_G+\mu_S+\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}}{2\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}} \\ \frac{\lambda_S}{\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}} & \frac{-\lambda_S+\mu_G-\mu_S+\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}}{2\sqrt{(\lambda_S+\mu_S+\mu_G)^2-4\mu_S\mu_G}} \end{pmatrix} \ . \tag{7.9}$$

Thus, we now have derived the components of $P(t)$ in (7.4). We have derived $P(t)$ as a nonlinear function of μ_G, λ_S and μ_S from (7.7)-(7.9).

7.2. The Associated Ergodic DTMC

We use the time-dependent transition matrix $P(t)$ in the role of (5.1). We now specify an updating time interval of length T. We then replace a dead fund by a good fund at time T. So we make an ergodic two-state DTMC with transition matrix

$$P = \begin{matrix} G \\ S \end{matrix} \begin{pmatrix} 1 - P_{G,S}(T) & P_{G,S}(T) \\ 1 - P_{S,S}(T) & P_{S,S}(T) \end{pmatrix} . \tag{7.10}$$

We construct P in (7.10) by letting $P_{G,S} = P_{G,S}(T)$ and $P_{S,S} = P_{S,S}(T)$ and then making the DTMC ergodic by letting the row sums be 1. In other words, we insert an instantaneous transition from state D to G at time T, which is the time of a single transition in the DTMC.

Paralleling (5.3), this two-state DTMC has steady-state probability vector π, where

$$\pi \equiv (\pi_G, \pi_S) = \left(\frac{1 - P_{S,S}(T)}{1 - P_{S,S}(T) + P_{G,S}(T)}, \frac{P_{G,S}(T)}{1 - P_{S,S}(T) + P_{G,S}(T)} \right) . \tag{7.11}$$

7.3. Parameter Fitting in the CTMC Model

We now proceed toward parameter fitting for this new model. Paralleling (5.4), we have the time-dependent death rate being

$$\delta \equiv \delta(T) = \pi_G\{1 - P_{G,G}(T) - P_{G,S}(T)\} + \pi_S\{1 - P_{S,G}(T) - P_{S,S}(T)\} . \tag{7.12}$$

Just as for the DTMC, we can derive the survival probability from the CTMC model, which is closely related to the death rate. At time t, the survival probability of a fund is defined as $S(t) = P_{G,G}(t) + P_{G,S}(t)$ for $t \geq 0$. Figure 13 displays the survival probabilities for the CTMC model with $\delta = 0.03, 0.06$ and 0.09. The survival probability for $\delta = 0.09$ is possible only in the CTMC model, since for the DTMC model, r becomes negative when $\delta \approx 0.07$. As we see from Figure 13, median fund life is less than 10 year for $\delta = 0.09$. Since this median hedge fund life is within the range of Rouah (2006) and Park (2006), it should be worth considering $\delta = 0.09$ with the CTMC model.

In addition to equation (7.12), we will need analogs of equations (5.5) and (5.6). These will involve transitions in the DTMC over the time interval of length T. In particular, we obtain the new equations

$$\gamma_G \cdot Y_G = P_{G,G}(T) \cdot Y_G + P_{G,S}(T) \cdot Y_S + \{1 - P_{G,G}(T) - P_{G,S}(T)\}Y_D \tag{7.13}$$

and

$$\gamma_S \cdot Y_S = P_{S,G}(T) \cdot Y_G + P_{S,S}(T) \cdot Y_S + \{1 - P_{S,G}(T) - P_{S,S}(T)\}Y_D . \tag{7.14}$$

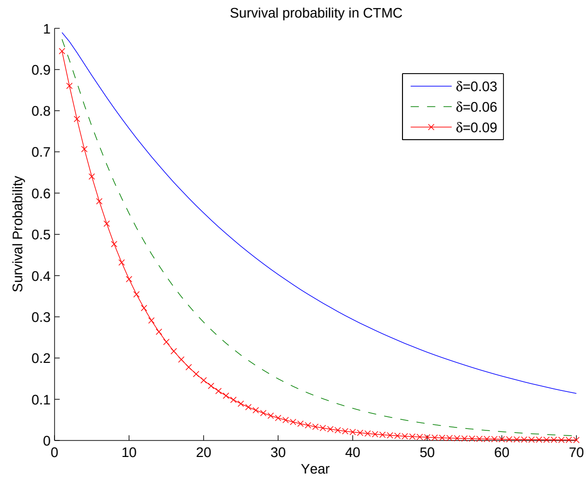


Figure 13: Survival probability curves for the CTMC model with $\delta = 0.03, 0.06$, and 0.09. The parameters values p, q, r, Y_G, Y_S, and Y_D are from table 1.

This is just as for the DTMC model before, except that we have to add the term for a D state when the fund starts in the G state at the beginning of the year. Since we are thinking of yearly updates, we let $T = 1$.

We now want to do the model fitting. We want to determine the three parameters μ_G, λ_S and μ_S, exploiting the three equations (7.12), (7.13) and (7.14), but we have been unable to obtain explicit solutions for the desired parameters as we did in §5.3. So we use an iterative algorithm.

We start with a candidate initial parameter triple $(\mu_G, \lambda_S, \mu_S)$. Given that parameter triple and the specified time T, we calculate the transition probabilities $P_{G,G}(T)$, $P_{G,S}(T)$, $P_{S,G}(T)$, and $P_{S,S}(T)$ in (7.4)–(7.6) by calculating the eigenvalues and eigenvectors of the infinitesimal matrix Q in (7.1). Afterwards we calculate the steady-state probability vector $\pi \equiv (\pi_G, \pi_S)$ in (7.11) of the two-state DTMC in (7.10). We then calculate the right-hand sides of the three equations (7.12)–(7.14). Our goal is to have three bonafide equations, where the two sides of the equations are equal, but in the iteration we will not achieve that. Based on the errors we see, we update the parameter triple $(\mu_G, \lambda_S, \mu_S)$ and repeat until the errors in the three equations (7.12)–(7.14) are negligible.

Since we are confronted with a three-dimensional iteration, we do not want to proceed in a haphazard way. Hence, we apply nonlinear programming to do this iteration. The idea is to find parameter triple (λ_S, μ_S, μ_G) minimizing errors between the right-hand and left-hand sides of equations (7.12), (7.13) and (7.14). To formulate a minimization problem, we define three error functions ϵ_1, ϵ_2 and ϵ_3 as a function of parameter triple $(\lambda_S, \mu_S, \mu_G)$ as follows:

$$\begin{aligned}
\epsilon_1 &\equiv \epsilon_1(\lambda_S, \mu_S, \mu_G) = \delta(T) - \pi_G\{1 - P_{G,G}(T) - P_{G,S}(T)\} - \pi_S\{1 - P_{S,G}(T) - P_{S,S}(T)\},\\
\epsilon_2 &\equiv \epsilon_2(\lambda_S, \mu_S, \mu_G) = \gamma_G \cdot Y_G - P_{G,G}(T) \cdot Y_G - P_{G,S}(T) \cdot Y_S - \{1 - P_{G,G}(T) - P_{G,S}(T)\}Y_D,\\
\epsilon_3 &\equiv \epsilon_3(\lambda_S, \mu_S, \mu_G) = \gamma_S \cdot Y_S - P_{S,G}(T) \cdot Y_G - P_{S,S}(T) \cdot Y_S - \{1 - P_{S,G}(T) - P_{S,S}(T)\}Y_D.
\end{aligned} \tag{7.15}$$

Our objective, then, is to find λ_S, μ_S and μ_G such that $\epsilon_1(\lambda_S, \mu_S, \mu_G) = \epsilon_2(\lambda_S, \mu_S, \mu_G) = \epsilon_3(\lambda_S, \mu_S, \mu_G) = 0$. To obtain values of ϵ_1, ϵ_2, and ϵ_3 for a given parameter triple of λ_S, μ_S and μ_G, we have to calculate $P_{G,G}(T), P_{G,S}(T), P_{S,G}(T)$, which are elements of $P(t)$ matrix in (7.4). As indicated above, this involves finding eigenvalues and eigenvectors of Q matrix in (7.1). From (7.6), we derived eigenvalues as a function of λ_S, μ_S and μ_G. Given the eigenvalues, the eigenvectors can be calculated as in (7.8), but also in other ways. Since Q is only a 2×2 matrix, calculation of the eigenvectors for given eigenvalues can be done easily. One way is to use the Schur decomposition algorithm, as in Anderson et al. (1999), which is implemented in MATLAB as the *eig* function. Then $\Lambda(t)$ can be calculated easily from (7.3), so we can easily compute the U and Λ matrices numerically. The final step is to compute $P_{G,G}(T)$, $P_{G,S}(T)$ and $P_{S,G}(T)$ from $P(t) = U\Lambda(t)U^{-1}$.

We can obtain the desired parameter triple $(\lambda_S, \mu_S, \mu_G)$ by solving the following constrained minimization problem:

$$\begin{aligned}
&\min_{\lambda_S, \mu_S, \mu_G} \quad \max\{|\epsilon_1|, |\epsilon_2|, |\epsilon_3|\}\\
&\textit{such that}\\
&\epsilon_1 = \delta(T) - \pi_G(1 - P_{G,G}(T) - P_{G,S}(T)) - \pi_S\{(1 - P_{S,G}(T) - P_{S,S}(T)\},\\
&\epsilon_2 = \gamma_G \cdot Y_G - P_{G,G}(T) \cdot Y_G - P_{G,S}(T) \cdot Y_S - \{1 - P_{G,G}(T) - P_{G,S}(T)\}Y_D,\\
&\epsilon_3 = \gamma_S \cdot Y_S - P_{S,G}(T) \cdot Y_G - P_{S,S}(T) \cdot Y_S - \{1 - P_{S,G}(T) - P_{S,S}(T)\}Y_D\\
&\lambda_S, \mu_S, \mu_G \geq 0
\end{aligned} \tag{7.16}$$

We regard γ_G, γ_S, Y_G, Y_S and Y_D as given constants, and we regard the BD rates λ_S, μ_S and μ_G as the variables. Since the transition probabilities $P_{G,G}(T)$, $P_{G,S}(T)$ and $P_{S,G}(T)$ are functions of the BD rates λ_S, μ_S and μ_G through the eigenvalue and eigenvector calculation, we must regard (7.16) as a nonlinear programming (NLP) problem, for which it is natural to apply an iterative procedure. However, since we only have three variables, we are able to solve the NLP (7.16) easily. One effective way is to use Sequential Quadratic Programming (SQP), as in Schittkowski (1986). With SQP, at each iteration, an approximation is made of the Hessian of the Lagrangian function using a quasi-Newton updating method. That is then used to generate a QP subproblem whose solution is used to form a search direction for a line search procedure. This algorithm is implemented in MATLAB via the functions *fminsearch* and *fmincon*. Both functions solve (7.16) within seconds.

In addition to fitting λ_S, μ_S and μ_G, we want to calibrate σ^2. To do so, we need to adjust the definition of σ^2 for the CTMC model. Suppose that π_G and π_S are the stationary probabilities for the transition matrix in (7.10). We let π'_D be the stationary probability that the fund dies at the end of 1 year when it starts alive before. This is equal to death rate δ in our definition:

$$\pi'_D = \delta = \pi_G\{1 - P_{G,G}(1) - P_{G,S}(1)\} + \pi_S\{1 - P_{S,G}(1) - P_{S,S}(1)\} \tag{7.17}$$

We then also define π'_G and π'_S accordingly, using (7.11):

$$\begin{aligned} \pi'_G &= \pi_G \cdot P_{G,G}(1) + \pi_S \cdot P_{S,G}(1) \\ \pi'_S &= \pi_G \cdot P_{G,S}(1) + \pi_S \cdot P_{S,S}(1) \end{aligned} \tag{7.18}$$

where (π_G, π_S) is defined in (7.11). Finally, the variance satisfies

$$\sigma^2 = \pi'_G \cdot Y_G^2 + \pi'_S \cdot Y_S^2 + \pi'_D \cdot Y_D^2 \ , \tag{7.19}$$

where π' is defined in (7.17) and (7.18). It turns out that we can easily achieve any desired σ, such as $\sigma \approx 0.1$, by iterating Y_G. Given (7.19), this iteration step is essentially the same as for the DTMC in §5.4.

Below are parameter values obtained using the NLP in (7.16) and iterating Y_G values. In the following table, ϵ records the maximum absolute value of errors in equations (7.12), (7.13) and (7.14). As before, we let $T = 1$.

Table 2: Parameter value sets for the CTMC model with $\gamma_G = \gamma_S = 0.5$

δ	μ_G	λ_S	μ_S	Y_G	Y_S	Y_D	Calculated σ	ϵ
0.00	0.2410	0.4791	0.0000	0.0670	-0.15	-0.20	0.1003	2.3033×10^{-6}
0.03	0.2204	0.5531	0.1240	0.0690	-0.15	-0.20	0.1005	3.9441×10^{-8}
0.06	0.2264	0.7017	0.3488	0.0700	-0.15	-0.20	0.1001	1.2849×10^{-6}
0.07	0.2286	0.7916	0.4741	0.0701	-0.15	-0.20	0.0997	1.4160×10^{-6}
0.09	0.2381	1.1063	0.8806	0.0710	-0.15	-0.20	0.0997	5.9465×10^{-7}

Unlike the DTMC model, where the parameter r becomes negative if δ exceeds 0.07 for the base-case parameter values, for the CTMC we can fit the model to δ up to around 0.13. When $\delta \approx 0.13$, we observe that the CTMC lockup premium becomes nearly 0.

7.4. The Lockup Premium Calculation

Once we have fit all the parameters, we can calculate the lockup premium. The procedure is essentially the same as in §5.6. For a 1-year Lockup, the fund's annual return is

$$R_1 = P_{G,G}(1) \cdot Y_G + P_{G,S}(1) \cdot Y_S + \{1 - P_{G,G}(1) - P_{G,S}(1)\} Y_D = \gamma_G \cdot Y_G \tag{7.20}$$

The fund's expected return after the i^{th} year if the fund is under lockup is

$$\begin{aligned} R_i &= P_{G,G}(i) \cdot Y_G + P_{G,S}(i) \cdot Y_S \\ &\quad + [P_{G,G}(i-1) \cdot \{1 - P_{G,G}(1) - P_{G,S}(1)\} + P_{G,S}(i-1) \cdot \{1 - P_{S,G}(1) - P_{S,S}(1)\}] \cdot Y_D \\ &\quad + \{1 - P_{G,G}(i-1) - P_{G,S}(i-1)\} \cdot \gamma_G \cdot Y_G. \end{aligned} \tag{7.21}$$

Just as for the DTMC in §5.6, the cumulative difference of expected returns between a 1-year lockup and an n-year lockup is

$$C_n = \sum_{i=1}^{n} (R_1 - R_i) = nR_1 - \sum_{i=1}^{n} R_i \tag{7.22}$$

The lockup premium is then the average difference $A_n = C_n/n$. Figure 3 in §1 shows the lockup premium functions for four different values of δ, ranging from 0.00 to 0.09. The remaining parameter values are as specified in Table 2.

7.5. Premium Comparison between CTMC and DTMC

Having created both the DTMC and CTMC models, it is interesting to see how they compare. Since funds die more quickly in the CTMC model, we expect that the lockup premium for the CTMC to be lower than for the DTMC model, and that is what we see. There is no difference at all for $\delta = 0$, but as δ increases the difference between the estimated lockup premiums increases. We provide several plots and tables in §G in the Appendix. Here we illustrate by showing the larger differences for $\delta = 0.07$ in Table 3 below.

Table 3: Lockup premium comparisons when $\delta = 0.07$

Lockup Period	2	3	4	5	6
DTMC	0.8822	1.1288	1.2215	1.2546	1.2593
CTMC	0.6428	0.9296	1.0642	1.1270	1.1528
(DTMC-CTMC)/DTMC (%)	27.1320	17.6480	12.8814	10.1705	8.4604

8. Conclusion

We have defined the hedge fund lockup premium as the average difference (per year) between the annual returns from investments in hedge funds, where one has a nominal one-year lockup and the other has an extended n-year lockup. We have developed DTMC and CTMC models to estimate the hedge-fund lockup premium as a function of the length n of the extended lockup period. To account for immediate redemption of investment when a hedge fund fails, we include a death state in the model. The lockup premium represents the cost of not being able to switch from sick funds to good funds while under the lockup condition. The effect of the lockup is mitigated by the death rate, and so is more difficult to analyze.

We have shown how the Markov chain models can be fit to basic hedge-fund performance measures, notably, the persistence of returns, γ (also allowing different γ_G and γ_S), the standard deviation of returns, σ, and the hedge-fund death rate δ. We then have applied the models to estimate how the lockup premium depends on these important performance measures. The models quantify how the lockup premium increases as a function of the persistence factor γ and the standard deviation σ, but decreases as a function of the death rate δ.

We examined the literature to see what researchers have concluded about hedge-fund performance persistence and the other hedge-fund performance measures, but we have found varying conclusions. We also performed our own statistical analysis using the TASS hedge fund data to estimate these hedge fund performance measures. We found strong evidence of persistence, but the specific persistence values cannot be predicted with great confidence, as is evident from the scatter plots in Figure 2. Thus we think we have been more successful showing how the lockup premium depends on the hedge-fund performance measures than in determining these performance measures themselves.

In §4 we provided a simple analysis without Markov chains to quantify the lockup premium in the case of no death. That analysis yields the explicit no-death lockup premium formulas in (4.4) and (4.7). In that case the lockup premium tends to be proportional to σ. In all cases, the lockup premium is a concave function of n, initially increasing and then eventually decreasing for $\delta > 0$ because the fund will eventually die, so that the lockup will eventually provide no extra penalty. The simple approximation with $\delta = 0$ yields an upper bound.

The model fitting requires solving equations. For the DTMC, we were able to give explicit formulas for the three DTMC parameters p, q and r as a function of Y_G, Y_S, Y_D, γ_G and γ_S, but in order to calibrate the standard deviation of returns, σ, we needed to use an iterative method.

For the CTMC we used a more involved iterative method based on nonlinear programming. For both models, we developed efficient algorithms for doing the model fitting.

We conclude that all three performance measures - δ, γ and σ - can have a significant impact on the lockup premium, but we predict that the effect will be negligible if either γ or σ is small. We estimated these key hedge-fund performance measures from the TASS database, but further work needs to be done to obtain reliable estimates.

The CTMC model is more realistic because the DTMC model requires two years for a transition from G to D. The CTMC model allows a wider range of δ - up to 0.13 instead of only up to 0.07 for the DTMC model - for the base case of parameters. Figure 3 shows for the CTMC model that the lockup premium for $\delta = 0.09$ is about half what it would be with $\delta = 0.00$.

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References

Agarwal, V., and N. Naik. 2000. Multi-Period Performance Persistence Analysis of Hedge Funds. *Journal of Financial and Quantitative Analysis* 35 (3): 327–342.

Agarwal, V., and N. Naik. 2005. Hedge Funds. *Foundations and Trends in Finance* 1 (2): 103–169.

Amihud, Y., and H. Mendelson. 1986. Asset Pricing and the Bid-ask Spread. *Journal of Financial Economics* 17 (2): 223–249.

Amihud, Y., and H. Mendelson. 1991. Liquidity, Maturity, and the Yields on U.S. Treasury Securities. *The Journal of Finance* 46 (4): 1411–1425.

Anderson, E., Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, and D. Sorensen. 1999. *Lapack user's guide.* Third Edition, SIAM, Philadelphia.

Aragon, G. 2007. Share restrictions and asset pricing: evidence from the hedge fund industry. *Journal of Financial Economics* 83 (1): 33–58.

Baquero, G., J. ter Horst, and M. Verbeek. 2005. Survival, look-ahead bias, and persistence in hedge fund performance. *Journal of Financial and Quantitative Analysis* 40 (3): 493–517.

Boyson, N., and M. Cooper. 2004. Do Hedge Funds Exhibit Performance Persistence? A New Approach. *Working Paper*, Northeastern and Purdue University.

Brown, S. J., W. Goetzmann, and R. Ibbotson. 1999. Offshore Hedge Funds Survival and Performance 1989-1995. *Journal of Business* 72 (1): 91–117.

Brown, S. J., W. Goetzmann, and J. Park. 2001. Careers and survival: competition and risk in the hedge fund and CTA industry. *Journal of Finance* 56 (5): 1869–1886.

Browne, S., and W. Whitt. 1995. Pricewise-Linear Diffusion Process. In *Advances in Queuing*, ed. J. H. Dshalalow, Probability and Stochastic Series, Chapter 18, 463–480. CRC Press.

Browne, S. J., M. Milevsky, and T. Salisbury. 2003. Asset Allocation and the Liquidity Premium for Illiquid Annuities. *The Journal of Risk and Insurance* 70 (3): 509–526.

Chordia, T., R. Roll, and A. Subrahmanyam. 2001. Market Liquidity and Trading Activity. *Journal of Finance* 56 (2): 501–530.

Darar, V., N. Y. Naik, and R. Radcliffe. 1998. Liquidity and Asset Returns: An Alternative Test. *Journal of Financial Markets* 1:203–219.

Derman, E. 2006. A Simple Model for The Expected Premium for Hedge Fund Lockups. *Working paper*, To appear in *Journal of Investment Management.*

Edwards, F., and M. Caglayan. 2001. Hedge Fund Performance and Manager Skill. *Journal of Futures Markets* 21 (11): 1003–1028.

Eleswarapu, V., and M. Reinganum. 1993. The Seasonal Behavior of Liquidity Premium in Asset Pricing. *Journal of Financial Economics* 34 (3): 373–386.

Fung, W., and D. A. Hsieh. 2000. Performance Characteristics of Hedge Fund and Commodity Funds: Natural Versus Spurious Biases. *Journal of Financial and Quantitative Analysis* 35 (3): 291–307.

Gregoriou, G. N. 2002. Hedge fund survival lifetimes. *Journal of Asset Management* 3 (3): 237–252.

Hayes, B. T. 2006. Maximum Drawdowns of Hedge Funds with Serial Correlation. *Journal of Alternative Investments* Spring: 26–38.

Ippolito, R. 1989. Efficiency with costly information: a study of mutual fund performance. *Quarterly Journal of Economics* 104 (1): 1–23.

Jagannathan, R., A. Malakhov, and D. Nonikov. 2006. Do Hot Hands Persist Among Hedge Fund Managers. NBER Working paper, 12015.

Karlin, S., and H. M. Taylor. 1975. *A first course in stochastic processes.* Academy Press.

Koh, F., W. Koh, and M. Teo. 2003. Asian Hedge Funds: Return Persistence Style and Fund Characteristics. *Working Paper*, Singapore management University.

Krishnamurthy, A. 2002. The Bond / Old-Bond Spread. *Journal of Financial Economics* 66 (2-3): 463–506.

Liang, B. 1999. On the Performance of Hedge Fund. *Financial Analyst Journal* 55 (4): 72–85.

Longstaff, F. 1995. How Much Can Marketability Affect Security Values ? *Journal of Finance* 50 (5): 1767–1774.

Longstaff, F. 2001. Optimal Portfolio Choice and the valuation of Illiquid Securities. *Review of Financial Studies* 14 (2): 407–431.

Longstaff, F. 2004. The Flight-to-Liquidity Premium in US Treasury Bond Price. *Journal of Business* 77 (3): 511–526.

Park, H. 2006. *Risk measures for hedge funds and a survival analysis.* Ph. D. thesis, University of Massachusetts.

Pastor, L., and R. Stambaugh. 2003. Liquidity Risk and Expected Stock Returns. *Journal of Political Economy* 111 (3): 642–685.

Roll, R. 1984. A Simple Implicit Measure of the Effective Bid-Ask Spread in an Efficient Market. *Journal of Finance* 34 (4): 1127–1139.

Ross, S. M. 2003. *Introduction to probability models*. Eighth ed. Academy Press.

Rouah, F. 2006. *Competing risks in hedge fund survival.* Ph. D. thesis, McGill University.

Schittkowski, K. 1986. NLQPL: A FORTRAN-Subroutine Solving Constrained Nonlinear Programming Problems. *Annals of Operations Research* 5 (2): 485–500.

ter Horst, J., T. Nijman, and M. Verbeek. 2001. Eliminating look-ahead bias in evaluating persistence in mutual fund performance. *Journal of Empirical Finance* 8 (4): 345–373.

Vayanos, D. 2004. Flight to Quality, Flight to Liquidity, and the Pricing of Risk. NBER working paper No. W10327.

Warga, A. 1992. Bond returns, Liquidity, and Missing Data. *Journal of Financial and Quantitative Analysis* 27 (4): 605–617.

Appendix

A. Overview

In this appendix we elaborate on several issues investigated in the main paper. We start with statistical issues. In §B we discuss our statistical analysis of the TASS database. We specify in detail how we selected the data to consider. In §C we describe an alternative ratio method for estimating the persistence of hedge fund returns.

We next turn to the DTMC models. In §D we consider the two-state DTMC model without dying funds, which provides a link between §4 and 5 in the main paper. We supplement §6 in the main paper in §E by providing additional descriptions of the way the three-state DTMC model parameters and the lockup premium depend on basic hedge fund performance measures.

We then turn to the CTMC model. In §F we supplement the nonlinear program algorithm (NLP) for fitting the CTMC parameters in §7.3 of the main paper by doing numerical experiments to verify that the NLP solution is unique. We present additional results comparing the lockup premiums for the DTMC and CTMC models in §G.

Finally, we consider some variants of the analysis procedures reported in the main paper. In §H we consider an alternative DTMC model with finer transition times, which approaches the CTMC model as the transition times get shorter. In §I we consider a different comparison, comparing a one-year lockup to a rolling n-year lockup model, where we assume that a dead fund under an n-year lockup is replaced by another fund with an n-year lockup, instead of a 1-year lockup, whenever reinvestment is required.

B. Hedge Fund Data Analysis

In this section, we describe the data analysis procedure to estimate the persistence of hedge fund annual returns. We first explain how we try to remove biases in our analysis. We then describe the regression procedure to estimate the persistence factor.

B.1. Data Selection

The data we use for our persistence analysis come from the TASS database. The TASS database archives monthly returns and the Net Asset Value (NAV) managed by the funds for each month and each hedge fund. In addition, TASS also archives fund-specific data such as the strategy category of a fund, the first date of reporting to TASS, the first date of operation, etc. The hedge funds in the TASS database are classified into eleven categories by strategy: convertible arbitrage, dedicated short bias, emerging markets, equity market neutral, event driven, fixed income arbitrage, fund of fund, global macro, long/short equity, managed futures, and others (or multiple strategies).

A fund usually keeps reporting its monthly returns as long as it continues operating. If a fund ceases reporting its returns to TASS, then the last date of the report is marked as the *drop date* in the data. A fund may stop reporting its returns if it is liquidated due to successive losses. However, it is not always true that a hedge fund suffers huge losses when it ceases reporting. In fact, even a successful fund may cease reporting if it does not want to reveal its performance publicly any more. Thus, we cannot simply count the number of funds dropped from the data to estimate the liquidation rate of hedge funds; see §3 in the main paper for further discussion. If the reason why a fund ceases reporting is available, TASS reports it, but it is often not given.

As mentioned above, TASS differentiates between the date the fund starts reporting and the date the fund starts operating. Thus, we can exclude one possible bias, which is so called *backfill bias*. When a fund starts reporting returns after operating for several months or years, the fund simultaneously reports several monthly returns at the time its first return is reported. It is then possible for the fund manager to drop or change some bad monthly returns which have been made before the reporting date, which may increase reported returns from the actual returns. Fung and Hsieh (2000) calculate that the difference from actual returns and reported returns is about 3.6% per year from this reason. Therefore, we consider monthly returns only after the fund's first reporting date. Similarly, if a fund's monthly returns are reported less than six times a year, we exclude these data due to the possibility of hiding or altering bad returns.

The other criterion we consider is the NAV managed by a fund. If a fund's managed assets are too small, then the monthly return might be too volatile, since it may have relatively less ability to diversify its risks. We assume that a fund has an ability to produce relatively stable returns once its managed assets reach a certain level. Specifically, we consider monthly returns only if the fund's NAV has reached 25 million dollars at least once, at which point we assume that the fund becomes mature, so that it can produce relatively stable returns. We remark that similar criteria were used by Boyson and Cooper (2004).

After selecting the monthly returns based on the above criteria, we make pairs of two successive annual returns for each hedge fund from 2000 to 2005. Thus, there are possibly six pairs of annual returns of a fund, if it does not cease reporting during that period. The monthly returns are annualized to measure the yearly persistence of returns. We use geometric compounding to convert twelve monthly returns to one annual return, i.e.,

$$r_{annual} = (1 + r_1) \cdot (1 + r_2) \cdots (1 + r_{12}) - 1 \ .$$

We next calculate relative annual returns for each fund by subtracting the average annual returns of the funds in the same strategy. The relative returns for two successive years are then coupled as a pair to estimate yearly persistence factor. In order to make meaningful pairs of relative returns for two successive years, the averages of annual returns for the first year and each strategy category of the funds are calculated first. When calculating the average annual returns and the associated relative returns for the next consecutive year, we only include returns from the funds which existed and were not dropped from the TASS database during the previous year. Thus, the average annual return for any given year depends on whether that year is treated as an initial year or a next year. They are not necessarily equal, since some funds may start reporting to TASS in the next consecutive year. In this way, we finally construct pairs of two consecutive relative returns from 2000 to 2005 for each strategy category of the fund.

Before conducting regression, we also exclude pairs of returns with extreme values, depending on the distribution of the pairs of returns for each strategy category. Even one or two outliers can seriously affect the regression, especially if we do not have a large number of observations. Specifically, we exclude pairs of relative returns when one absolute relative return exceeds $\pm$ 30 % for fixed income arbitrage, equity macro and $\pm$ 40% for convertible, dedicated short bias, and global macro strategy categories. We also exclude pairs of relative returns exceeding $\pm$ 50% for emerging market, event driven, fund of fund, long/short equity, managed future, and others strategy categories of funds.

We conduct a linear auto-regression analysis with pairs of two successive years of annual relative returns. The coefficient from this linear regression, i.e., the least square fit is the calculated persistence. The regression analysis results in very low intercept for all strategy category. Thus, we finally conduct a auto-regression without intercept and consider only the coefficient term.

B.2. Statistical Results

Table 4 shows the auto-regression results from the data we selected. We did the analysis for all the data as well as by strategy category. We observe that the regression result for all the strategy categories merged shows positive persistence factor of 0.22. The P-values give the probability of seeing the observed persistence if there actually were none. The last two columns show 95% confidence intervals for each persistence factor, i.e., the regression coefficient. We observe that the estimated persistence factors varies, but for most strategy categories, it is significantly high. Note that 9 out of 11 strategy categories statistically reject zero persistence hypothesis with 95%. Consequently, 9 out of 11 lower 95% estimation values are positive. However, the regression analysis shows that R^2 is very low. This implies that there is considerable randomness, so that we may not see persistence for individual funds. However, the P-values show that positive persistence for individual funds exists in the average.

Table 4: Auto-regression analysis results

strategy	number of observations	persistence γ	lower 95%	upper 95%	R^2	P-value
All	4788	0.22	0.19	0.25	0.05	4.79×10^{-60}
Convertible arbitrage	238	0.44	0.34	0.55	0.22	2.41×10^{-14}
Dedicated short bias	29	0.49	0.11	0.86	0.20	0.01
Emerging market	315	0.36	0.26	0.47	0.13	6.75×10^{-11}
Event driven	533	0.24	0.16	0.31	0.07	7.95×10^{-10}
Equity macro	268	0.09	-0.01	0.19	0.01	0.07
Fixed income arbitrage	193	0.29	0.15	0.42	0.08	5.27×10^{-5}
Fund of fund	986	0.33	0.28	0.38	0.14	4.94×10^{-33}
Global macro	166	0.13	-0.02	0.28	0.02	0.09
Long short equity	1658	0.15	0.11	0.19	0.03	2.01×10^{-11}
Managed future	235	0.20	0.08	0.32	0.04	1.46×10^{-3}
Other	167	0.48	0.32	0.64	0.18	1.33×10^{-8}

We next consider the standard deviation of the annual returns selected above. We display sample standard deviation for selected annual returns in Table 5 and observe that they are within the range we consider in this paper: from 0.05 to 0.25. For several strategy categories, the number of selected returns is too small to obtain meaningful estimates of the standard deviation. Thus, merging all returns from 2000 to 2004 for each strategy category may give better insight about the real variability of the annual returns.

Table 5: Estimated standard deviation of annual returns (%)

strategy	Number of observation	2000	2001	2002	2003	2004	All
All	4788	0.18	0.13	0.13	0.16	0.09	0.14
Convertible arbitrage	238	0.08	0.07	0.07	0.09	0.04	0.08
Dedicated short bias	29	0.02	0.16	0.12	0.19	0.14	0.18
Emerging market	315	0.19	0.21	0.17	0.23	0.15	0.24
Event driven	533	0.12	0.08	0.10	0.12	0.09	0.12
Equity macro	268	0.14	0.06	0.09	0.07	0.07	0.08
Fixed income arbitrage	193	0.07	0.04	0.09	0.06	0.06	0.07
Fund of fund	986	0.13	0.07	0.06	0.08	0.04	0.07
Global macro	166	0.07	0.11	0.11	0.14	0.07	0.11
Long short equity	17	0.22	0.16	0.15	0.17	0.10	0.18
Managed future	235	0.14	0.12	0.14	0.14	0.10	0.14
Other	167	0.18	0.13	0.13	0.16	0.09	0.14

C. Persistence from Ratios of Average Relative Returns

An alternative way to estimate the persistence factor is to consider the ratio of the next-year average returns to the current-year average return, restricting attention to the returns that are positive in the current year. Table 6 is the ratio of two successive average returns restricting attention to the returns that are positive and negative in the current year, respectively.

Since the average of relative returns is zero by definition, the ratio of averages for positive and negative returns should be identical, as observed. As can be seen from Table 4 and 6, these persistence estimates tend to be higher than the regression estimates.

D. The DTMC Model Without Death

We now return to the DTMC model and elaborate upon the analysis of the case $\delta = 0$. If we consider the DTMC without hedge funds dying, then we can work with a two-state DTMC, which has the transition matrix

$$P = \begin{matrix} G \\ S \end{matrix} \begin{pmatrix} p & 1-p \\ 1-r & r \end{pmatrix} , \qquad (4.1)$$

which has only the two parameters p and r.

Let $\pi \equiv (\pi_G, \pi_S)$ be the steady-state probability vector of the two-state DTMC with transition matrix in (4.1). A convenient explicit expression for π is

$$\pi \equiv (\pi_G, \pi_S) = \left(\frac{1-r}{(1-r)+(1-p)}, \frac{1-p}{(1-r)+(1-p)} \right) = \left(\frac{1-r}{2-r-p}, \frac{1-p}{2-r-p} \right) . \qquad (4.2)$$

Table 6: Ratio of average relative returns for good states

(i) For positive current relative returns

Strategy	current year average relative return	next year average relative return	ratio (γ)
Convertible	7.25	2.84	0.39
Dedicated Short Bias	16.40	7.14	0.44
Fixed Income	6.44	2.44	0.38
Fund of Fund	4.65	1.45	0.31
Others	10.09	3.86	0.38

(ii) For negative current relative returns

Strategy	current year average relative return	next year average relative return	ratio (γ)
Convertible	-4.83	-1.89	0.39
Dedicated Short Bias	-10.93	-4.76	0.44
Fixed Income	-6.57	-2.49	0.38
Fund of Fund	-4.41	-1.37	0.31
Others	-7.09	-2.71	0.38

Since Y_G and Y_S are the assumed relative returns (deviations from the mean return), we can express variance of the fund's performance in steady state as

$$\sigma^2 = \pi_G \cdot Y_G^2 + \pi_S \cdot Y_S^2 \ . \tag{4.3}$$

To satisfy, (4.3), we calibrate p and r in the transition matrix P of (4.1). We do this from two equations for persistence factor γ, expressed as a function of p and q.

$$\gamma \cdot Y_G = p \cdot Y_G + (1-p) \cdot Y_S \tag{4.4}$$

and

$$\gamma \cdot Y_S = (1-r) \cdot Y_G + r \cdot Y_S \ . \tag{4.5}$$

From (4.4) and (4.5), we derive that

$$p = \frac{\gamma \cdot Y_G - Y_S}{Y_G - Y_S}, \quad r = \frac{Y_G - \gamma \cdot Y_S}{Y_G - Y_S} \ . \tag{4.6}$$

From the above equations, it is straightforward to verify that

$$\frac{Y_G}{\sigma} \frac{-Y_S}{\sigma} = 1 \ . \tag{4.7}$$

If we fix $Y_S/\sigma = -1.5$, we have

$$\frac{Y_G}{\sigma} = \frac{1}{1.5} \approx 0.67 \ , \tag{4.8}$$

which exactly matches the analysis without Markov chain when $\delta = 0$ in §4 in the main paper. By assuming normally distributed annual relative returns, and letting Y_G be the median of the positive returns, in §4 we found that $Y_G/\sigma =$ median $\{|N(0,1)|\} = 0.67$.

Recall that the lockup premium formula in (5.17) of the main paper is

$$A_n \equiv \frac{C_n}{n} \equiv R_1 - \frac{1}{n}\sum_{i=1}^{n} R_i = \gamma Y_G - \frac{1}{n}\sum_{i=1}^{n}\left(P^i_{G,G}\cdot Y_G + P^i_{G,S}\cdot Y_S + P^i_{G,D}\cdot Y_D\right) \ .$$

Since $P^i_{G,D} = 0$ for all i when $\delta = 0$, we obtain

$$P^i_{G,G}\cdot Y_G + P^i_{G,S}\cdot Y_S = \left(P^{i-1}_{G,G}\cdot P_{G,G} + P^{i-1}_{G,S}\cdot P_{S,G}\right) Y_G + \left(P^{i-1}_{G,G}\cdot P_{G,S} + P^{i-1}_{G,S}\cdot P_{S,S}\right) Y_S \ . \quad (4.9)$$

Using (4.4) and (4.5), the formula above reduces to

$$P^i_{G,G}\cdot Y_G + P^i_{G,S}\cdot Y_S = \gamma\left(P^{i-1}_{G,G}\cdot Y_G + P^{i-1}_{G,S}\cdot Y_S\right) \ . \quad (4.10)$$

Then, since $P_{G,G}\cdot Y_G + P_{G,S}\cdot Y_S = \gamma Y_G$, from (4.10), the lockup premium formula becomes

$$A_n \equiv \frac{C_n}{n} \equiv R_1 - \frac{1}{n}\sum_{i=1}^{n} R_i = \gamma Y_G - \frac{1}{n}\sum_{i=1}^{n}\gamma^i Y_G \ , \quad (4.11)$$

Notice that becomes the same formula in (4.4) of §4 in the main paper, i.e.,

$$A_n = \gamma Y_G\left(1 - \frac{1-\gamma^n}{n\cdot(1-\gamma)}\right) \ .$$

E. Sensitivity Analysis for the DTMC Model

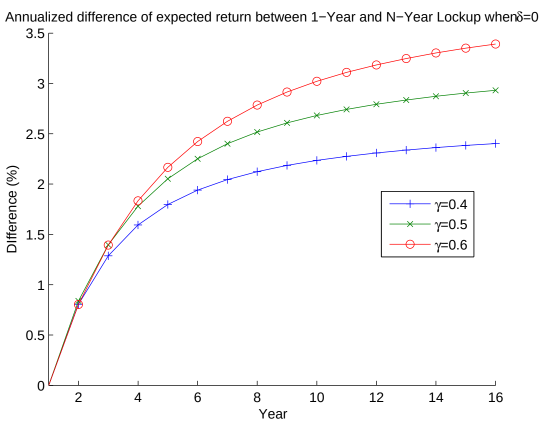
In this section we do more sensitivity analysis, expanding on the discussion in §6 in the main paper. We first carry out the calculations for the base case, using the parameter values derived in §5.3. Our model depends on three exogenous variables, δ, γ, σ. We at first emphasized how the lockup premium depends on the death rate δ. It is also important to investigate how the lockup premium depends upon γ and σ.

E.1. How the Lockup Premium Depends on γ (γ_G, γ_S) and σ

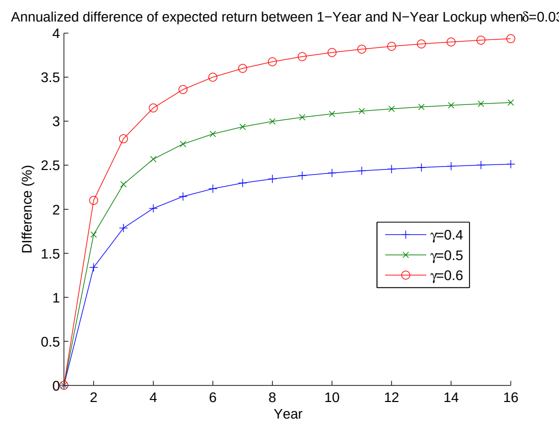
We now see how much the premium depends on the model parameters $\gamma(\gamma_G, \gamma_S)$ and σ. Table 7 shows the model parameters for $\gamma = 0.4, 0.5$, and 0.6 and figure 14 shows the lockup premium for $\gamma = 0.4, 0.5$, and 0.6. The figure suggests that as the persistence factor γ decreases, the n-year lockup premium decreases. The DTMC model works for persistence factor as low as

Table 7: Parameter value sets for γ ranging from 0.2 to 0.6

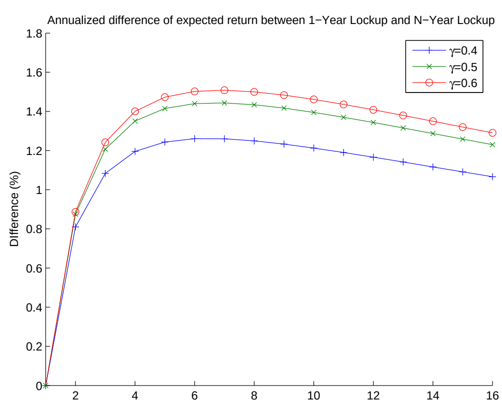
γ	δ	p	q	r	Y_G	Y_S	Y_D	Calculated σ
0.4	0.00	0.8147	0.4147	0.5853	0.067	-0.15	-0.20	0.1002
0.4	0.03	0.8147	0.4427	0.4360	0.067	-0.15	-0.20	0.1000
0.4	0.06	0.8147	0.4892	0.1877	0.067	-0.15	-0.20	0.0998
0.5	0.00	0.8456	0.3456	0.6544	0.067	-0.15	-0.20	0.1002
0.5	0.03	0.8432	0.3719	0.5030	0.0685	-0.15	-0.20	0.1001
0.5	0.06	0.8409	0.4207	0.2282	0.070	-0.15	-0.20	0.1001
0.6	0.00	0.8765	0.2765	0.7235	0.067	-0.15	-0.20	0.1002
0.6	0.03	0.8727	0.3029	0.5645	0.070	-0.15	-0.20	0.0997
0.6	0.06	0.8679	0.3590	0.2324	0.074	-0.15	-0.20	0.1002
0.2	0.06	0.7615	0.6298	0.0782	0.0637	-0.15	-0.20	0.1002
0.3	0.06	0.7879	0.5586	0.1374	0.0652	-0.15	-0.20	0.1000
0.4	0.06	0.8147	0.4892	0.1877	0.067	-0.15	-0.20	0.0998



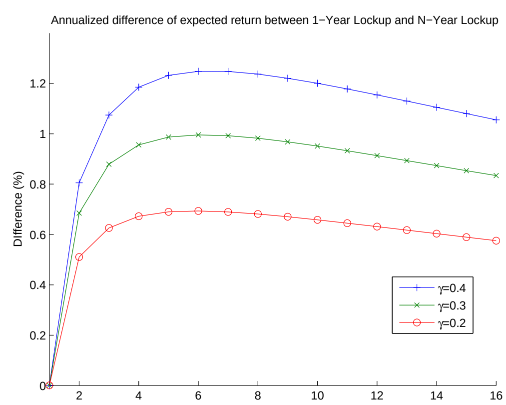
(i) With $\delta = 0$ for $\gamma = 0.4, 0.5, 0.6$



(ii) With $\delta = 0.03$ for $\gamma = 0.4, 0.5, 0.6$



(iii) With $\delta = 0.06$ and $\gamma = 0.4, 0.5, 0.6$



(iv) With $\delta = 0.06$ and $\gamma = 0.2, 0.3, 0.4$

Figure 14: The lockup premium for the DTMC model for parameter values in Tables 7.

0.1. However, the n-year lockup premium decreases to the amount lower than 0.5 % for any n. Figure 14 suggests that the estimation of γ is important, especially for small δ and large n.

We next consider two separate persistence factors, γ_G and γ_S in Table 8 and the sensitivity of the lockup premium with respect to γ_G and γ_S. Note that in the third line of Table 8, r is negative, which breaks down the DTMC model. Figure 15 shows the lockup premium for parameters in Table 8.

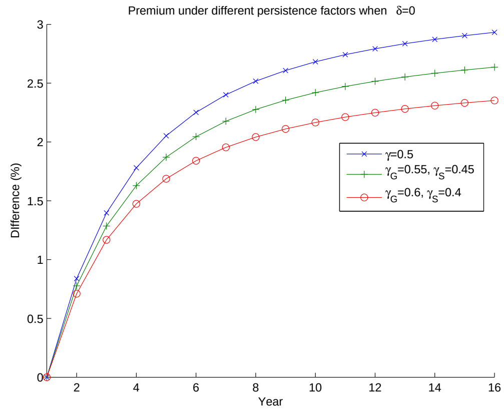
Table 8: Parameter value sets for γ_G and γ_S

γ_G	γ_S	δ	p	q	r	Y_G	Y_S	Y_D	Calculated σ
0.6	0.4	0.00	0.8655	0.3982	0.6018	0.076	-0.15	-0.20	0.1000
0.6	0.4	0.03	0.8643	0.4320	0.4069	0.077	-0.15	-0.20	0.1002
0.6	0.4	0.06	0.8637	0.5068	-0.0127	0.775	-0.15	-0.20	0.1000
0.55	0.45	0.00	0.8547	0.3725	0.6275	0.0715	-0.15	-0.20	0.1000
0.55	0.45	0.03	0.8527	0.4014	0.4583	0.0730	-0.15	-0.20	0.1002
0.55	0.45	0.06	0.8513	0.4603	0.1276	0.074	-0.15	-0.20	0.1001
0.5	0.5	0.00	0.8456	0.3456	0.6544	0.067	-0.15	-0.20	0.1002
0.5	0.5	0.03	0.8432	0.3719	0.5030	0.0685	-0.15	-0.20	0.1001
0.5	0.5	0.06	0.8409	0.4207	0.2282	0.070	-0.15	-0.20	0.1001

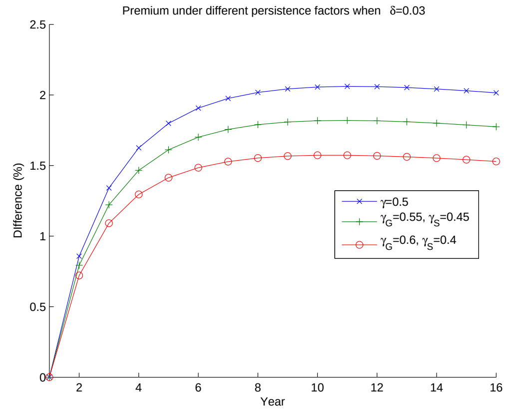
We lastly check the sensitivity of the lockup premium with respect to σ. Our TASS database analysis estimates σ of annual returns for each year is lower than 0.20 in most cases. We here highlight the sensitivity of the lockup premium for $\sigma = 0.05, 0.10$, and 0.15 with $\gamma = 0.5$. Table 9 is the parameter value sets and Figure 16 is the corresponding lockup premium. We see that the premium increases in σ.

Table 9: Parameter value sets for $\sigma = 0.05, 0.10, 0.15$

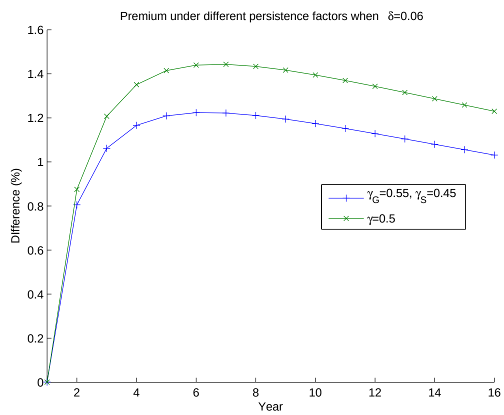
σ	δ	p	q	r	Y_G	$Y_S = -1.5\sigma$	$Y_D = -2.0\sigma$	σ (calculated)
0.05	0.00	0.8461	0.3461	0.6539	0.0333	-0.075	-0.10	0.05
0.10	0.00	0.8461	0.3461	0.6539	0.0667	-0.150	-0.20	0.10
0.15	0.00	0.8461	0.3461	0.6539	0.1000	-0.225	-0.30	0.15
0.05	0.03	0.8434	0.3721	0.5025	0.0342	-0.075	-0.10	0.05
0.10	0.03	0.8434	0.3721	0.5025	0.0684	-0.150	-0.20	0.10
0.15	0.03	0.8434	0.3721	0.5025	0.1026	-0.225	-0.30	0.15
0.05	0.06	0.8410	0.4209	0.2275	0.0350	-0.075	-0.10	0.05
0.10	0.06	0.8410	0.4209	0.2275	0.0699	-0.150	-0.20	0.10
0.15	0.06	0.8410	0.4209	0.2275	0.1049	-0.225	-0.30	0.15



(i) With $\delta = 0$



(ii) With $\delta = 0.03$



(iii) With $\delta = 0.06$

Figure 15: The lockup premium for the DTMC with $\gamma_G \neq \gamma_S$.

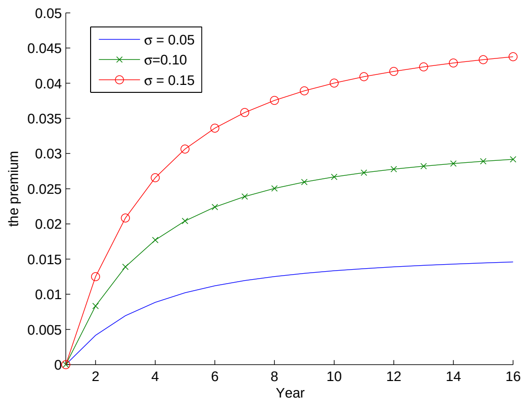
E.2. Sensitivity of $p, q,$ and r with respect to δ, Y_G, Y_S, and Y_D

In this section, we observe the effect of δ to the implied transition probabilities p, q, r. Figure 8 is the implied transition probabilities for δ from 0 to 0.1. It is clear that the transition probability from sick to sick state, r, is the most sensitive to δ. Simple calculation of the partial derivative of r with respect to δ shows that

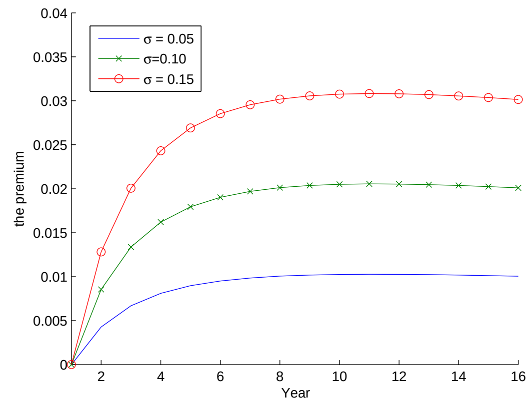
$$\frac{\partial r}{\partial \delta} = -\frac{\frac{2-p-r}{1-p}(Y_G - Y_D)}{(Y_G - Y_S) - \frac{\delta}{1-p}(Y_D - Y_D)} \,. \tag{5.1}$$

From (5.1), we observe that as δ increases, r decreases more rapidly. Note that the coefficient of δ in (5.1) is $(Y_G - Y_D)/(1-p) \approx 1$ thus its impact is big. We also observe that r becomes negative as δ increases above 0.07. Thus, the maximum allowable death rate in the DTMC model is 0.07. Fortunately, this restriction is relaxed in the CTMC model.

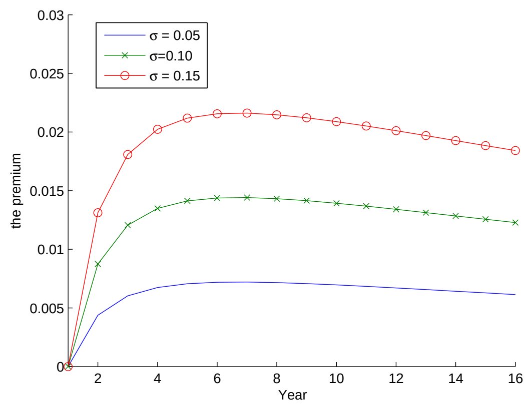
The implied transition probabilities are calculated for Y_G, Y_S, and Y_D in Figure 11 in §6. The plots show that $p, q,$ and r are sensitive to Y_G, but that there is even more dependence



(i) With $\delta = 0.00$



(ii) With $\delta = 0.03$



(iii) With $\delta = 0.06$

Figure 16: The lockup premium for the DTMC for $\sigma = 0.05, 0.10$, and 0.15.

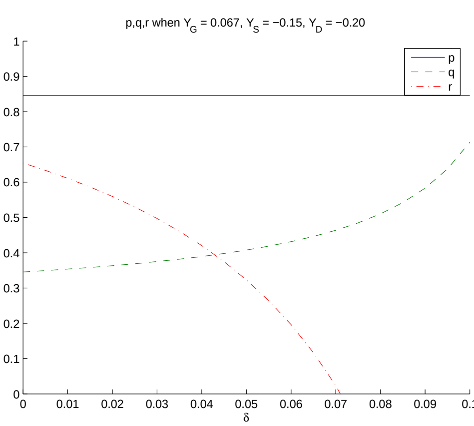


Figure 17: Implied transition probabilities p, q, r for δ from 0 to 0.1 with $Y_G = 0.067, Y_S = -0.15, Y_D = -0.20$ and $\gamma = 0.5$

upon γ, especially when $\gamma > 0.75$. The sensitivity of p, q, and r to Y_S and Y_D is much less, as is shown in Figure 11. This justifies that our parameter fitting method which changes Y_G

while fixing Y_G and Y_D since Y_G has greater effect to the implied transition probabilities than Y_S and Y_D. Notice that from (5.8), p is independent of Y_D and a linear function of γ.

We next consider the sensitivity of the steady-state probabilities π_G and π_S to the model parameters. Up until the critical point in γ, the steady-state probabilities π_G and π_S are less sensitive to γ, Y_S, and Y_D, but is sensitive to Y_G, which can be regarded as a function of σ, as illustrated in Figure 18.

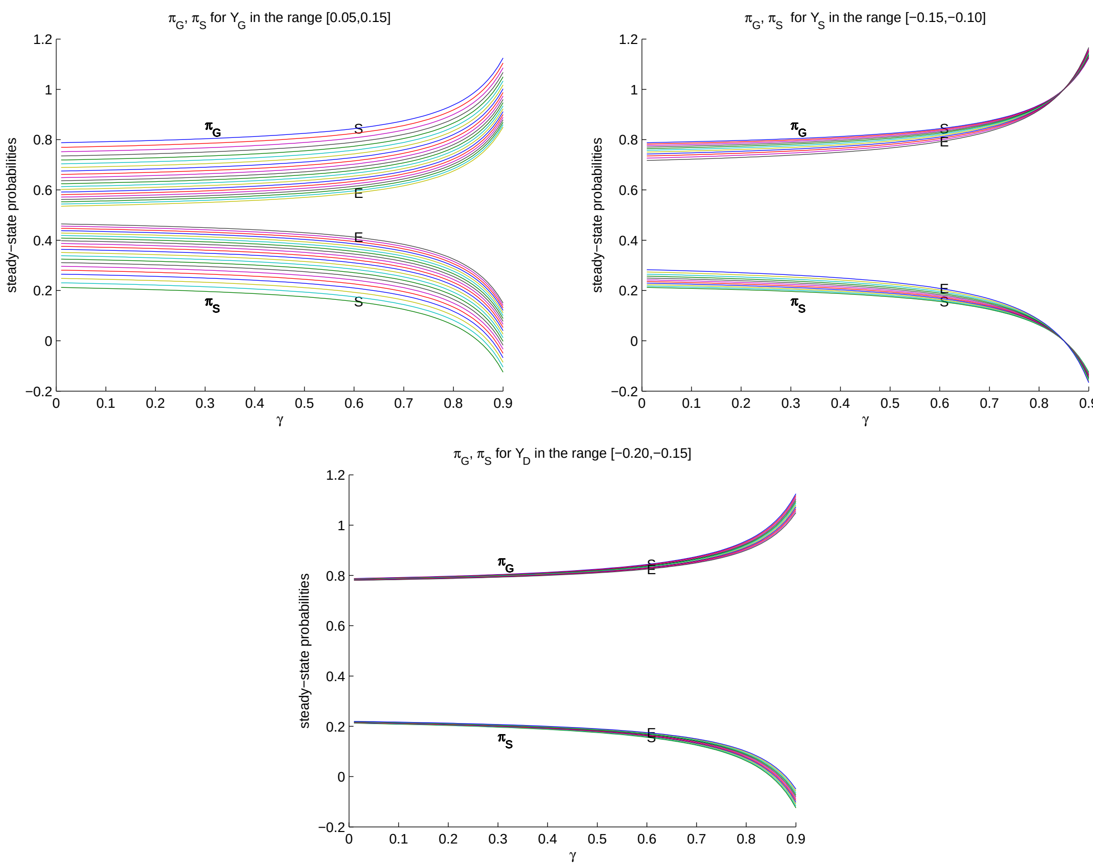


Figure 18: The steady-state probabilities π_G and π_S as a function of γ in the base case for values of Y_G ranging from 0.05 (starting value, denoted by S) to 0.15 (ending value, denoted by E), Y_S ranging from -0.15 to -0.10, and Y_D ranging from -0.20 to -0.15.

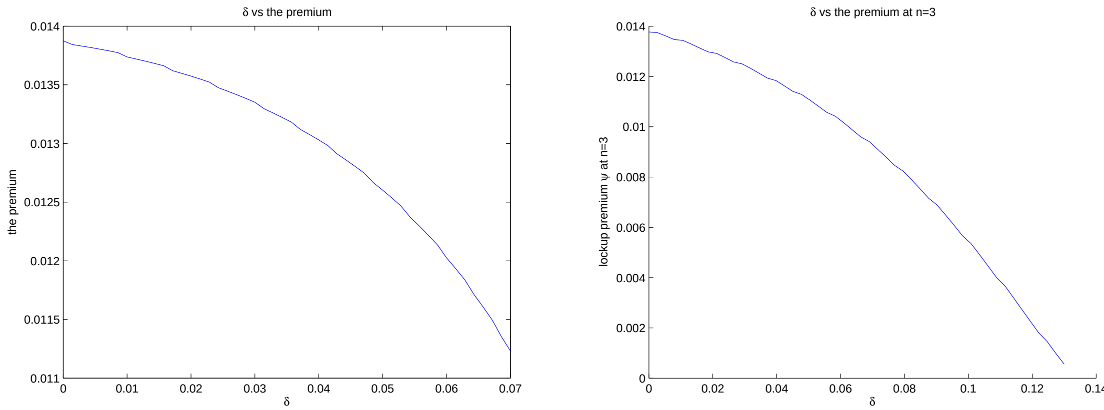
E.3. Sensitivity of the Premium for a Fixed Lockup Period

In this section, we investigate how the lockup premium for a fixed lockup period depends on the three variables δ, γ, and σ. We consider $n = 3$ with the choice of parameters $Y_S/\sigma = -1.5$ and $Y_D/\sigma = -2.0$. The result is helpful to estimate the lockup premium for a fixed lockup period when some of the three variables change. From Figures 4 and 10, it is clear that the lockup premium increases as δ decreases, γ increases, or σ increases. It turns out that the three-year lockup premium can be approximated reasonably well by a simple linear function of each variable separately, at least over a narrow range.

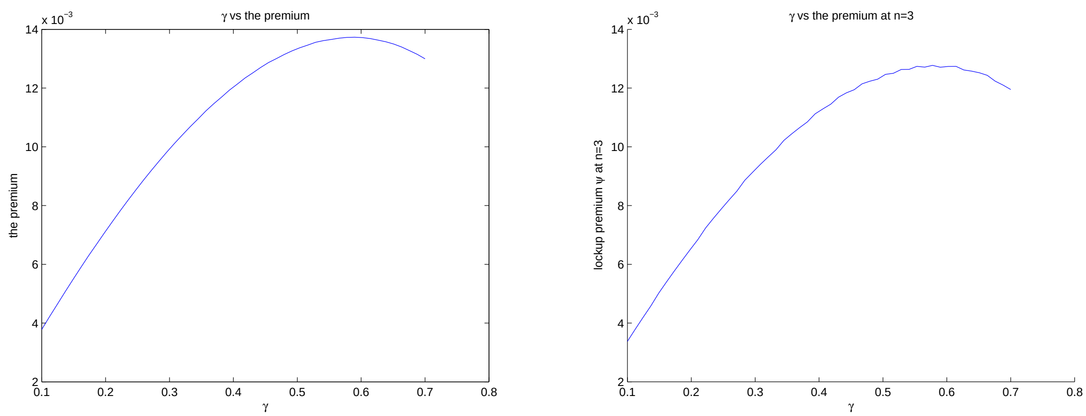
Figures 19 shows how the three-year lockup premium depends on each variable for the DTMC and CTMC models. Figure 19 (i) and (ii) show that the three-year lockup premium is only a weakly concave function of δ for the DTMC and CTMC models.

Figures 19 (iii) and (iv) show the three-year lockup premium as a function of the persistence factor γ for the DTMC and CTMC models. We see that the three-year lockup premium is a concave function of γ. That property was observed for the no-death lockup premium formula in (4.4). From (4.4), it is straightforward to verify that A_n is Y_G times a concave function of γ when $\delta = 0$. However, for $\gamma \leq 0.5$, the three-year lockup premium is only weakly concave function of γ.

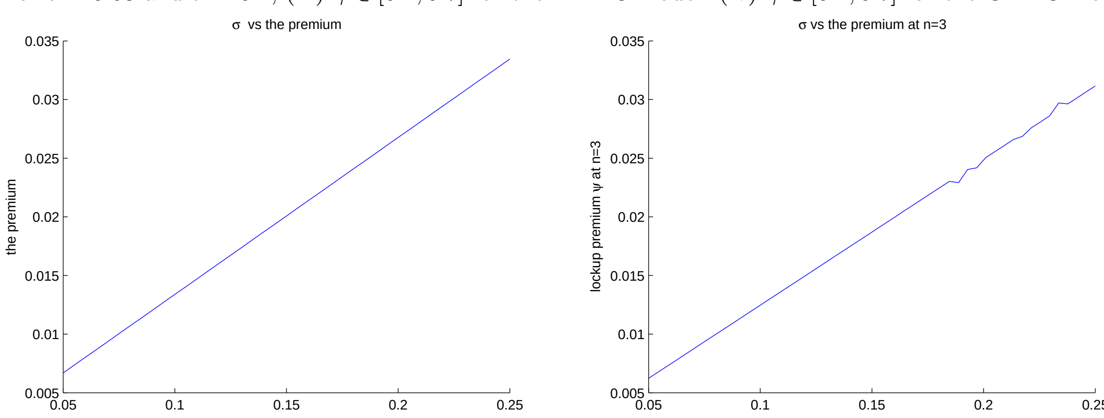
Figures 19 (v) and (vi) show the three-year lockup premium as a function of σ. We observe that the three-year lockup premium is very nearly a linear function of σ. In all cases, we see that the three-year lockup premium is only weakly concave or linear function for each of the three variables. Thus, a linear estimation of the three-year lockup premium for a relatively narrow range of the variables should work reasonably well.



For $\gamma = 0.5$ and $\sigma = 0.1$, (i) $\delta \in [0.0, 0.07]$ for the DTMC model (ii) $\delta \in [0.00, 0.13]$ for the CTMC model



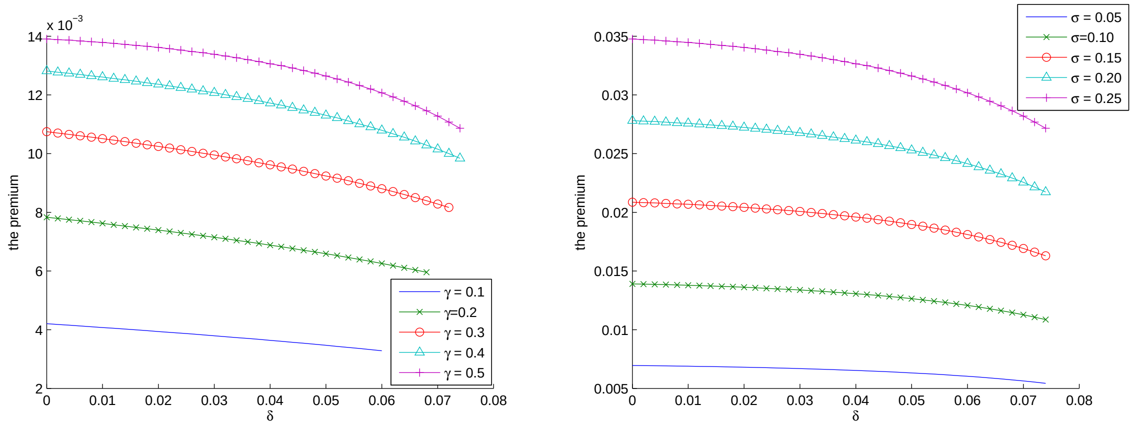
For $\delta = 0.03$ and $\sigma = 0.1$, (iii) $\gamma \in [0.1, 0.7]$ for the DTMC model (iv) $\gamma \in [0.1, 0.7]$ for the CTMC model



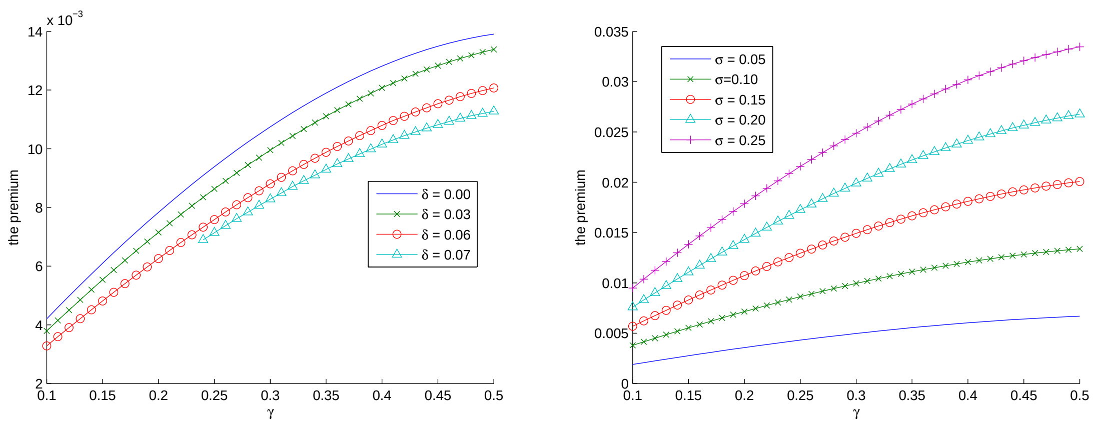
For $\delta = 0.03$ and $\gamma = 0.5$, (v) $\sigma \in [0.05, 0.25]$ for the DTMC model (vi) $\sigma \in [0.05, 0.25]$ for the CTMC model

Figure 19: The three-year lockup premium with one persistence factor $\gamma(= \gamma_G = \gamma_S)$ and $Y_S = -0.15, Y_D = -0.20$

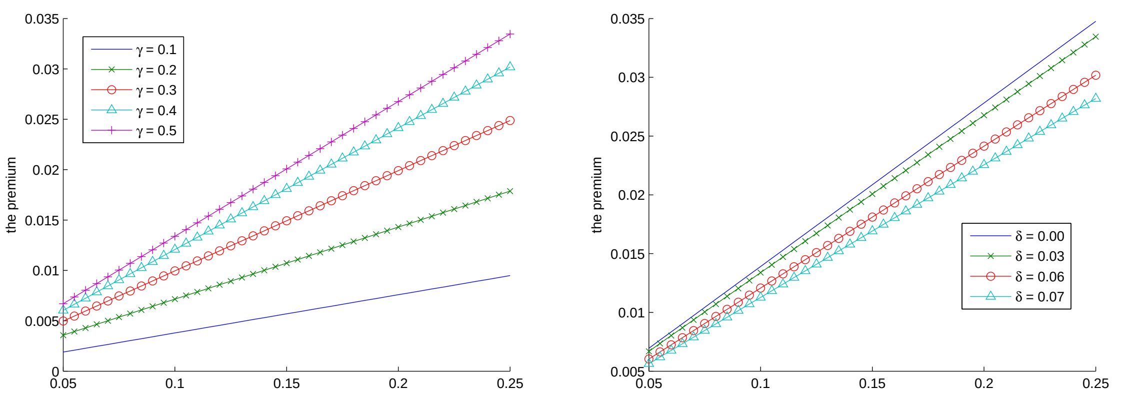
Figure 20 shows the three-year lockup premium as a function of σ, γ, and δ for the DTMC models. In this figure, we see how the three-year lockup premium depends on two of the three variables σ, γ, and δ while fixing the remaining one variable. For example, Figure 20 (i) shows the change of the three-year lockup premium for δ and γ while fixing σ as 0.1. We observe the near-perfect linearity of the three-year lockup premium for σ. We also observe that the concavity of the premium for γ increases as γ increases. Figures 20 (i) and (ii) suggest that the three-year lockup premium is quite insensitive to δ, which implies that the effect of δ on the lockup premium is relatively small. However, we caution the reader that the predicted impact of δ on the lockup premium is greater for the CTMC model, as shown in Figure 3 in §1 of the main paper. Also, for a longer lockup period than three year, the effect of δ on the lockup premium generally increases, which is shown in Figure 4 in the main paper. Figure 20 (iii) and (iv) show the three-year lockup premium for $\delta = 0.03$. To supplement that, Figure 21 illustrates how the three-year lockup premium for different γ and σ changes with $\delta = 0.00$ and 0.06. We observe that the shape of the three-year lockup premium function does not change as δ changes.



For $\delta = 0.00$ to 0.08, (i) $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$ with $\sigma = 0.1$ (ii) $\sigma = 0.05, 0.1, 0.15, 0.2, 0.25$ with $\gamma = 0.5$

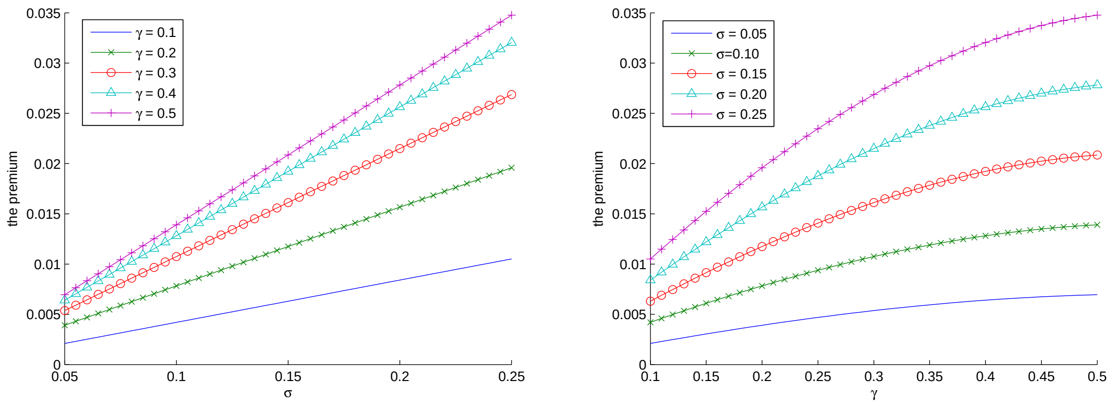


For $\gamma = 0.1$ to 0.5, (iii) $\delta = 0.00, 0.03, 0.06, 0.07$ with $\sigma = 0.1$ (iv) $\sigma = 0.05, 0.1, 0.15, 0.2, 0.25$ with $\delta = 0.03$

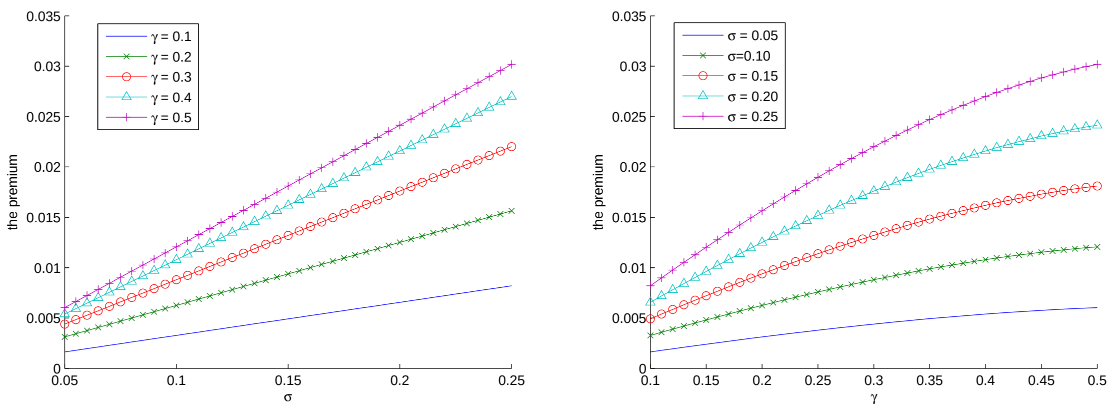


For $\sigma = 0.05$ to 0.25, (v) $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$ with $\delta = 0.03$ (vi) $\delta = 0.00, 0.03, 0.06, 0.07$ with $\gamma = 0.5$

Figure 20: The three-year lockup premium for the DTMC model with $Y_S = -1.5\sigma, Y_D = -2.0\sigma$. The lockup premium does not exist if q or r becomes negative.



For $\delta = 0.00$, (i) $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$ (ii) $\sigma = 0.05, 0.1, 0.15, 0.2, 0.25$



For $\delta = 0.06$, (iii) $\sigma = 0.05, 0.1, 0.15, 0.2, 0.25$ (iv) $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$

Figure 21: The three-year lockup premium for the DTMC model with $Y_S = -1.5\sigma, Y_D = -2.0\sigma$.

E.4. Estimating the Functional Form of the Three-year Lockup Premium

So far, we have calculated the lockup premium for variables γ, σ, and δ with the DTMC model. Since we have calculated the premium as a function of three variables, it is then natural to consider a simple functional form to describe the premium. If the estimation can be done relatively easily, it is useful to approximate the premium with a closed-form expression of three variables, denoted by $\psi(\delta, \gamma, \sigma)$. (Again, we fix $Y_S/\sigma = -1.5$ and $Y_D/\sigma = -2.0$.) We may then understand the effect of these three variables more intuitively. It is also easy to quickly estimate how the premium changes if the variables change.

The three-year lockup period is interesting because this is the first year a fund starting in a good state may becomes dead in the DTMC model. Thus, we can see the effect of the death of a fund on the lockup premium. Furthermore, three years is a practical case to consider. Recall that for $\delta = 0$ case, we have simple analytic formula for the lockup premium. Thus, we consider an estimation of three-year lockup premium as a closed form expression of γ, σ, and δ. However, the approximation also works for different lockup period and the choice of Y_S/σ. (See the remark below.)

Figure 20 suggests that the three-year lockup premium is weakly concave function of γ, linear function of σ and relatively insensitive to δ. We thus try a simple product form of three variables with an exponent for each variable. Specifically, denoting the three-year lockup premium as a function of δ, γ, and σ by $\psi_{(3)}(\delta, \gamma, \sigma)$, we consider the following simple candidate approximation:

$$\psi_{(3)}(\delta, \gamma, \sigma) \approx \psi^p_{(3)}(\delta, \gamma, \sigma) \equiv a\ \delta^b \gamma^c \sigma^d\ . \tag{5.2}$$

Taking logarithms of both sides of (5.2), it is straightforward to estimate the parameters a, b, c, and d from the calculated three-year lockup-premium values with linear regression, because

$$\ln \psi^p_{(3)}(\delta, \gamma, \sigma) = \ln a + b \ln \delta + c \ln \gamma + d \ln \sigma\ . \tag{5.3}$$

Since $\lim_{\delta \to 0} \psi^p_{(3)} = \infty$ when $b < 0$ and 0 when $b > 0$, which is not desirable for our estimation purpose, we have to restrict range of δ away from 0. Thus, we restrict the range of δ to [0.01.0.07].

It turns out that without further restricting the ranges of the variables δ, γ and σ, the candidate function $\psi^p_{(3)}(\delta, \gamma, \sigma)$ approximates the three-year lockup premium reasonably well. For example, for $\delta \in [0.01, 0.07]$, the linear regression of (5.3) approximates the three-year lockup premium by

$$\psi^p_{(3)}(\delta, \gamma, \sigma) = 0.15\ \delta^{-0.11} \gamma^{0.74} \sigma^{1.00},$$

with maximum error of 0.0039. Notice that the exponent to δ is -0.11, which eventually makes $\lim_{\delta \to 0} \psi^p_{(3)}(\delta, \gamma, \sigma) = \infty$. Thus, we expect that as δ approaches 0, e.g., for $\delta \ll 0.01$, the estimation function will not approximate the three-year lockup premium in DTMC model well. Figure 23 shows the estimation of the three-year lockup premium with the function obtained from the regression above for the selected ranges of variables. We observe that the estimation function approximates the three-year lockup premium reasonably well. If we further restrict the ranges of the variables such that $\gamma \in [0.2, 0.4]$, the maximum error reduces to 0.0017, which is only less than 6.7% of the three-year lockup-premium values in the DTMC model.

It is interesting to compare $\psi^p_{(3)}(\delta, \gamma, \sigma)$ to the lockup premium formula with $\delta = 0$ in §4 of the main paper. If we define $h(\gamma) \equiv \left(2\gamma - \gamma^2 - \gamma^3\right)/3$, the formula of 3-year lockup premium in that section is expressed as

$$A_3(\gamma, \sigma) = Y_G h(\gamma)\ .$$

We calibrate $Y_G = c\ \sigma$ by defining Y_G as the mean ($c = 0.8$) or median ($c = 0.67$) of normal random returns in §4 of the main paper. In particular, $c = 0.67$ coincides with the lockup premium formula in the DTMC model with $\delta = 0$ where we set $Y_S = -1.5\sigma$ and $Y_D = -2.0\sigma$.

We find that 0.74 is the best exponent approximating $h(\gamma)$ in the range of $\gamma \in [0.1, 0.5]$ from the regression. In other words, $b = 0.74$ is the best estimation from the regression of $a + b \ln \gamma = \ln(2\gamma - \gamma^2 - \gamma^3)/3$. Figure 22 shows that the direct approximation works well, especially when γ is less than 0.4. This partly explains why the estimation function $\psi^p_{(3)}$ works better when we restrict range of γ to $[0.2, 0.4]$.

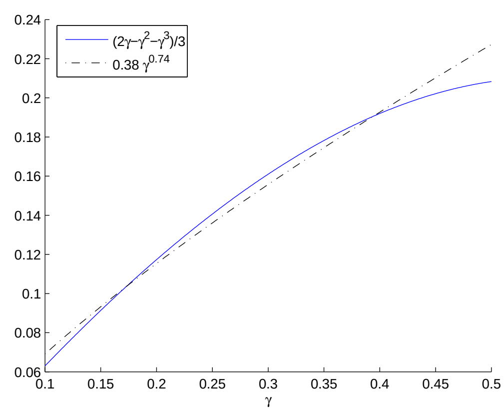
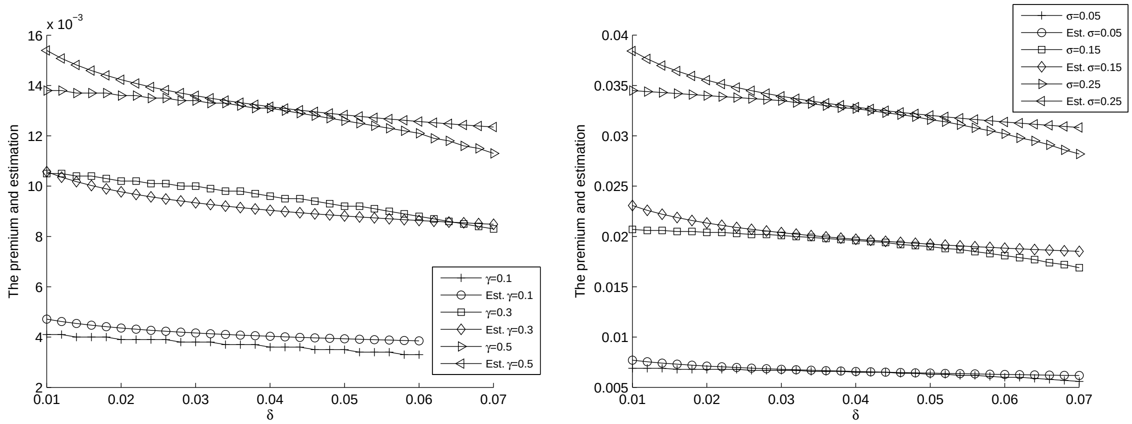


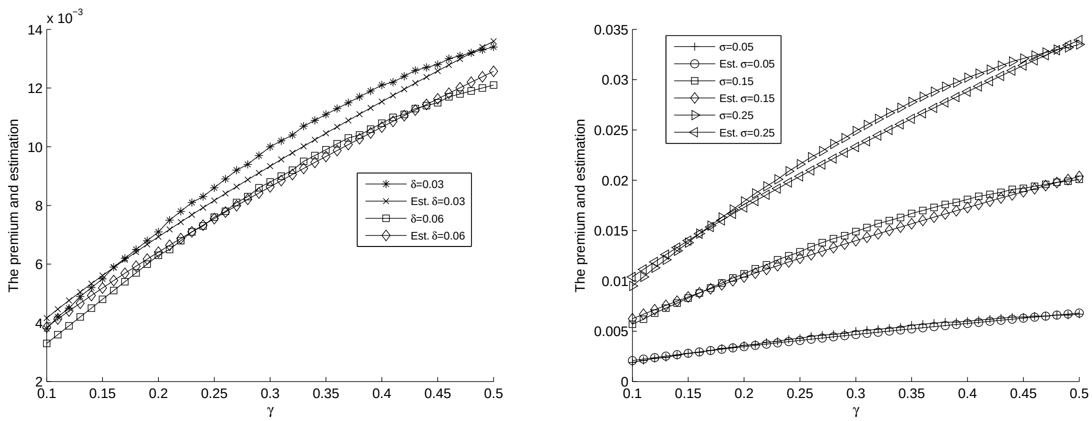
Figure 22: Approximation of $h(\gamma) \equiv (2\gamma - \gamma^2 - \gamma^3)/3$ by $0.38\gamma^{0.74}$

Since $h(\gamma)$ is approximated by $0.38\gamma^{0.74}$ and both the $A_3(\gamma, \sigma)$ and $\psi^p_{(3)}(\delta, \gamma, \sigma)$ are directly proportional to σ, the only difference between the two formula is the term associated with $\delta^{-0.11}$ in $\psi^p_{(3)}(\delta, \gamma, \sigma)$. This suggests that we can approximate the dependence of the three-year lockup premium on δ by a simple exponent term of $\delta^{-0.11}$. Also, we observe that the other relationships between the three-year lockup and γ and σ remain almost unchanged from $\delta = 0$ case. We see that $\delta^{-0.11}$ explains the weak decreasing relationship of the three-year lockup on δ well.

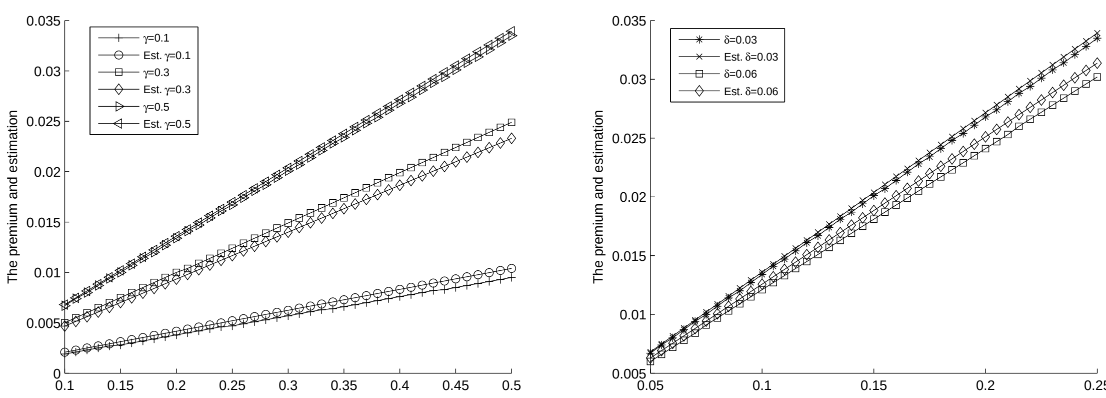
Finally, we remark that the approximation of the fixed-year lockup premium in the DTMC model by a product function $\psi^p(\delta, \gamma, \sigma) = a\ \delta^b \gamma^c \sigma^d$ works reasonably well for the other lockup periods (n) and other choices of Y_S/σ. For example, the four-year lockup premium in the DTMC model is approximated by $\psi^p_{(4)}(\delta, \gamma, \sigma) = 0.16\ \delta^{-0.24}\gamma^{0.76}\sigma^{1.00}$ with maximum error of 0.0071. If we choose $Y_S/\sigma = -1.0$, $\psi^p_{(3)}(\delta, \gamma, \sigma) = 0.16\ \delta^{-0.18}\gamma^{0.69}\sigma^{1.00}$ approximates the three-year lockup premium with maximum error of 0.0068, which can be reduced to 0.0028 if we restrict the range of γ, requiring that it be less than 0.4. The approximation also holds reasonably well if we change n and Y_S/σ at the same time, although the maximum error increases slightly. As before, for $\delta \in [0.01, 0.07]$, if we choose $Y_S/\sigma = -1.0$, the four-year lockup premium in the DTMC model is approximated by $\psi^p_{(4)}(\delta, \gamma, \sigma) = 0.16\ \delta^{-0.24}\gamma^{0.76}\sigma^{1.00}$ with maximum error of 0.0101. Again, the error reduces to 0.0031 if we further restrict the range of γ, requiring that it be less than 0.4.



For $\delta = 0.01$ to 0.07, (i) $\gamma = 0.1, 0.3, 0.5$ with $\sigma = 0.1$ (ii) $\sigma = 0.05, 0.15, 0.25$ with $\gamma = 0.5$



For $\gamma = 0.1$ to 0.5, (iii) $\delta = 0.03, 0.06$ with $\sigma = 0.1$ (iv) $\sigma = 0.05, 0.15, 0.25$ with $\delta = 0.03$



For $\sigma = 0.05$ to 0.25, (v) $\gamma = 0.1, 0.3, 0.5$ with $\delta = 0.03$ (vi) $\delta = 0.03, 0.06$ with $\gamma = 0.5$

Figure 23: Evaluating the product approximation $\psi^{p}_{(3)}(\delta, \gamma, \sigma) = 0.15\ \delta^{-0.11}\gamma^{0.74}\sigma^{1.00}$ for the three-year lockup premium

F. Testing Uniqueness for the Solution of the NLP

We use a nonlinear program (NLP) to fit the CTMC model parameters in §7.3. We now investigate whether or not the solution to the NLP is unique. To test for uniqueness of the solution to the NLP, we impose an additional constraint that forces us to calculate the solution outside the range of the original solution. We then compare the optimal solution of the new NLP and the error in that optimal solution to the original solution and its error.

For example, when $\delta = 0.03$, our parameter values are $\mu_G = 0.2205, \lambda_S = 0.5531, \mu_{=}$0.1240. We then recalculate optimal solution of the (7.16) with one additional condition of $\mu_G \geq 0.2205 + e$, where e is a small positive number. That clearly disallows the previous optimal solution. For $\delta = 0.03$, we observe that the optimal errors (ϵ) increase significantly when we impose such a constraint upon any of the three variables. For $e = 0.1$, the computed error (ϵ) increases from about 10^{-6} to more than 10^{-3}, as can be seen from the numerical results in Table 10. In addition, we see that the new constraint is always binding in the optimal solution, because the new optimal parameter value is the same as the new constraint value. (Here we do not iterate parameter Y_G to match σ.)

Table 10: Uniqueness test for $\delta = 0.03$ and 0.06 with e=0.1

(i) For $\delta = 0.03$

New Constraint	new μ_G	new λ_S	new μ_S	ϵ
$\mu_G \geq 0.2204 + e$	0.3204	0.7183	0.1551	0.0108
$\mu_G \leq 0.2204 - e$	0.1204	0.4288	0.1537	0.0931
$\lambda_S \geq 0.5531 + e$	0.1834	0.6531	0.2089	0.0060
$\lambda_S \leq 0.5531 - e$	0.1652	0.4531	0.0961	0.0080
$\mu_S \geq 0.1240 + e$	0.1790	0.6688	0.2240	0.0067
$\mu_S \leq 0.1240 - e$	0.3388	0.3680	0.0240	0.0189

(ii) For $\delta = 0.06$

New Constraint	new μ_G	new λ_S	new μ_S	ϵ
$\mu_G \geq 0.2264 + e$	0.3264	0.8717	0.3458	0.0105
$\mu_G \leq 0.2264 - e$	0.1264	0.5864	0.4453	0.0134
$\lambda_S \geq 0.7017 + e$	0.2096	0.8017	0.4539	0.0029
$\lambda_S \leq 0.7017 - e$	0.2523	0.6017	0.2507	0.0043
$\mu_S \geq 0.3488 + e$	0.2103	0.7968	0.4488	0.0028
$\mu_S \leq 0.3488 - e$	0.2530	0.5997	0.2488	0.0044

We observe the same effect for smaller e, such as *0.05* or even *0.01*. In each case, the new constraint is binding in the optimal solution. Those results strongly suggest that there is only one optimal solution for each parameter set. We observe similar results for $\delta = 0.06$, as shown in Table 10.

G. CTMC Versus DTMC

In this section we elaborate on the discussion in §7.5 of the main paper, where we discuss comparisons between the DTMC and CTMC models. We observed that the lockup premium is less for the CTMC model.

In the DTMC model, a fund which starts in a good state can become dead only after its second year. In contrast, in the CTMC model, a good fund can become dead any time in continuous time. In this regard, the CTMC model is more realistic. Since all funds follow a 1-year lockup after death, we expect that the lockup premium for the CTMC to be lower than for the DTMC model. And that is what we see.

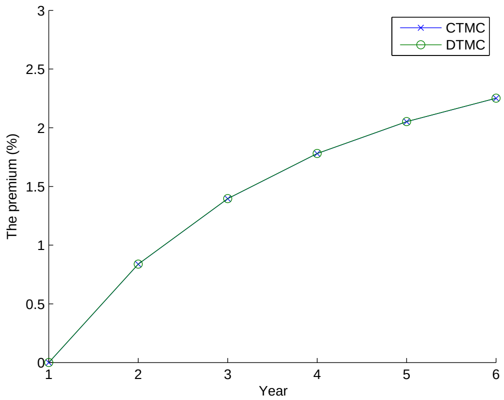
Specific differences are shown in Figure 24. As in our previous numerical examples, we fix the parameter values for several variables as: $\gamma_G = \gamma_S = \gamma = 0.5$, $\sigma = 0.1$, $Y_S = -0.15$ and $Y_D = -0.20$. The variable Y_G is calculated as part of the algorithm. As can be seen from Tables 1 and 2, the values for Y_G differ by very little.

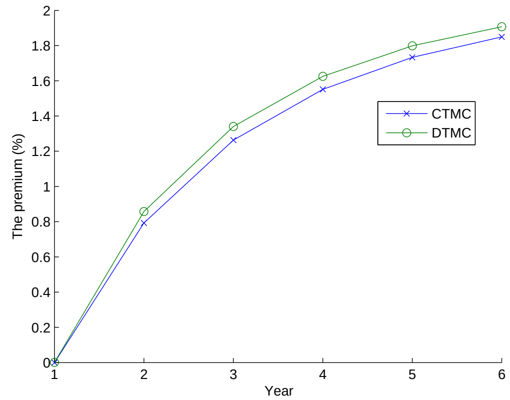
There is no difference at all between the lockup premiums when $\delta = 0$, as we should expect from §4, but the the differences between the DTMC and CTMC lockup premiums increase as δ increases.

Table 11 shows that the differences are significant for larger δ, reaching around 27 % if $\delta = 0.07$. As the death rate δ increases, the percentage difference between the DTMC and CTMC lockup premium predictions increases.

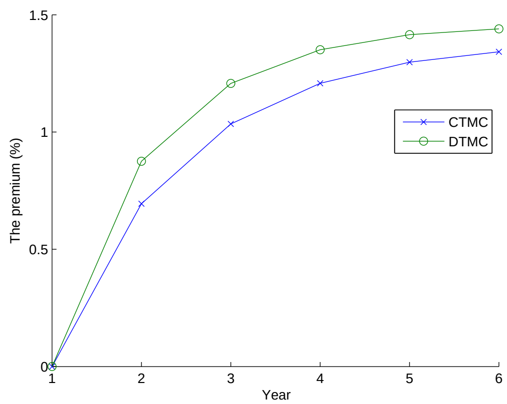
Table 11: Premium comparison between the DTMC and CTMC model

	Year				
(i) $\delta = 0.03$	2	3	4	5	6
DTMC	0.8575	1.3406	1.6251	1.7986	1.9070
CTMC	0.7932	1.2635	1.5517	1.7333	1.8498
(DTMC-CTMC)/DTMC (%)	7.50	5.75	4.51	3.63	3.00
(ii) $\delta = 0.06$					
DTMC	0.8752	1.2072	1.3509	1.4149	1.4397
CTMC	0.6943	1.0343	1.2080	1.2977	1.3419
(DTMC-CTMC)/DTMC (%)	20.68	14.38	10.58	8.28	6.79
(iii) $\delta = 0.07$					
DTMC	0.8822	1.1288	1.2215	1.2546	1.2593
CTMC	0.6428	0.9296	1.0642	1.1270	1.1528
(DTMC-CTMC)/DTMC (%)	27.13	17.65	12.88	10.17	8.46





(i) With $\delta = 0$ (ii) With $\delta = 0.03$



(iii) With $\delta = 0.06$

Figure 24: The lockup premium for the DTMC and CTMC model for $\delta = 0, 0.03, 0.06$.

H. The DTMC Model with Finer Transition Times

So far, we have assumed that the transitions occur once a year in the DTMC model. In this section, we assume that the transitions occur more frequently. We are mainly interested in the impact of shorter transition periods on the lockup premium. We want to see how the CTMC model is approached as the transition times become finer. Hereafter, we refer to the DTMC model with the $1/m$ year transition time by the $1/m$ DTMC model.

As mentioned in §3 of the main paper, some papers on the performance persistence of hedge funds suggest that there is considerable persistence over a shorter period than one year; e.g., over a quarter. Thus, we may estimate the persistence factor γ for a finer transition period with data and use it for the $1/m$ DTMC model. However, for simplicity, we show the sensitivity of the transition period to the lockup premium without re-calibration to data. In other words, we use the same yearly persistence for the $1/m$ DTMC model that we already used for $m = 1$.

A standard way to construct the transition matrix for the $1/m$ DTMC model from the transition matrix for the DTMC model is to divide the off-diagonal transition probabilities of the transition matrix in the DTMC model by m and adjust the diagonal probabilities so that the sum of the transition probabilities in each row is 1. Thus, the transition matrix for the $1/m$ DTMC model, denoted by P_m, is:

$$P_m = \begin{matrix} G \\ S \\ D \end{matrix} \begin{pmatrix} 1-(1-p)/m & (1-p)/m & 0 \\ q/m & 1-(1-r)/m & (1-q-r)/m \\ 0 & 0 & 1 \end{pmatrix} . \tag{8.1}$$

Even though we allow transition every $1/m$ years, we evaluate returns and make reinvestments every year. That closely parallels what we did for the CTMC model. Let $R_{m,i}$ denote the expected return of a fund in the i^{th} year in the $1/m$ DTMC model. The expected return in the first year is:

$$R_{m,1} \; = \; Y_G \cdot P^m_{m,(G,G)} + Y_S \cdot P^m_{m,(G,S)} + Y_D \cdot P^m_{m,(G,D)} \; . \tag{8.2}$$

Then,

$$\begin{aligned} R_{m,i} =& Y_G \cdot P^{i\cdot m}_{m,(G,G)} + Y_S \cdot P^{i\cdot m}_{m,(G,S)} \\ &+ Y_D[P^{(i-1)\cdot m}_{m,(G,G)} \cdot P^m_{m,(G,D)} + P^{(i-1)\cdot m}_{m,(G,S)} \cdot P^m_{m,(S,D)}] + R_{m,1} \cdot P^{(i-1)\cdot m}_{m,(G,D)} \end{aligned} \tag{8.3}$$

To explain (8.3), recall that a fund can die in one year even if it starts in a good state in the $1/m$ DTMC model. Thus, the transition probability of being dead given that the fund was in a good or sick state in the previous year is

$$\begin{aligned} &\mathbb{P}(\text{dead at the } i^{th} \text{ year } \mid \text{ good or sick at the } i-1^{th} \text{ year}) \\ &= P^{(i-1)\cdot m}_{m,(G,G)} \cdot P^m_{m,(G,D)} + P^{(i-1)\cdot m}_{m,(G,S)} \cdot P^m_{m,(S,D)}. \end{aligned} \tag{8.4}$$

Since $P^m_m \neq P$ in the DTMC model in general, $R_{m,1}$ in the equation (8.2) is not necessarily equal to $\gamma \cdot Y_G$ if we use the same parameter values in the DTMC model for the $1/m$ DTMC model without re-calibration. Our computations show that for $m = 20$ and $\delta = 0$, $\gamma \cdot Y_G = 0.0335$ whereas $R_{m,1} = 0.0404$. For $\delta = 0.03$, $\gamma \cdot Y_G = 0.0343$ whereas $R_{m,1} = 0.0411$. For $\delta = 0.06$, $\gamma \cdot Y_G = 0.0335$ and $R_{m,1} = 0.0390$. Thus, care must be taken to compare the 1-Year lockup premium and the n-year lockup premium. To be consistent in the comparison, we use $R_{m,1}$ for the expected return in the first year instead of $\gamma \cdot Y_G$.

The cumulative difference of expected returns between a 1-year lockup and an n-year lockup in the $1/m$ DTMC model is

$$
\begin{aligned}
C_{m,n} &= \sum_{i=1}^{n} \left(R_{m,1} - R_{m,i}\right) \\
&= \sum_{i=1}^{n} \Big[\Big(R_{m,1} \Big) - \Big(Y_G \cdot P_{m,(G,G)}^{i \cdot m} + Y_S \cdot P_{m,(G,S)}^{i \cdot m} \\
&\quad + Y_D [P_{m,(G,G)}^{(i-1)\cdot m} \cdot P_{m,(G,D)}^{m} + P_{m,(G,S)}^{(i-1)\cdot m} \cdot P_{m,(S,D)}^{m}] + R_{m,1} \cdot P_{m,(G,D)}^{(i-1)\cdot m} \Big) \Big]
\end{aligned} \tag{8.5}
$$

The lockup premium in the $1/m$ DTMC model is obtained by dividing $C_{m,n}$ by n, i.e.,

$$
A_{m,n} = \frac{1}{n} \sum_{i=1}^{n} \left(R_{m,1} - R_{m,i}\right) = \frac{1}{n} C_{m,n} \tag{8.6}
$$

Figure 25 shows the lockup premium for $\delta = 0, 0.03$, and 0.06 in the $1/m$ DTMC model. The impact of transition period on the lockup premium is greatest when m changes from 1 to 2.

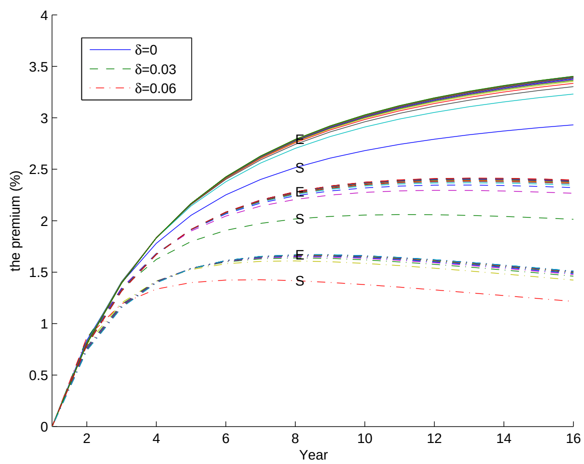


Figure 25: The lockup premium function for three values of the hedge fund death rate ($\delta = 0, 0.03, 0.06$) for different transition time periods ($1/m = 1, 1/2, \ldots, 1/10$). The starting (S) value is $m = 1$, whereas the ending (E) value is $m = 10$.

I. The Rolling Lockup Model

So far, we have assumed that a dead fund with an n-year lockup is replaced by a fund with a 1-year lockup. In this section, we assume instead that a dead fund is replaced by a fund having the same n-year lockup. We call this model the rolling lockup model. In reality, if a fund is dead before an n-year lockup ends, the investor may invest his redeemed money in a fund in any lockup period. The rolling lockup model is an alternative way to make the comparison.

In order to implement the rolling lockup model, we consider the age of good and sick funds. With an n-year lockup, the age of a fund can vary from 0 (new) to $n-1$ years. At n^{th} year, a fund is initialized and starts with a good state again in a new n-year lockup. For example, in a 3-year lockup, we assume that a fund is initialized to the new good state at the beginning of the third year. We denote a newly good fund by $G0$, a 1-year-old good fund by $G1$, and a 2-year-old good fund by $G2$. As for the sick state, we only need to consider $S1$ and $S2$ accordingly. The transition matrix for a 3-year lockup in the rolling lockup model is:

$$P_3 = \begin{matrix} G0 \\ G1 \\ G2 \\ S1 \\ S2 \end{matrix} \begin{pmatrix} 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 1 & 0 & 0 & 0 & 0 \\ 1-q-r & 0 & q & 0 & r \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9.1}$$

The third and fifth rows describe the initialization of a fund when the lockup period ends. In the fourth row, a 1-year-old sick fund ($S1$) becomes a newly good fund ($G0$) with probability $1-q-r$ if it dies at the end of the first year and then replaced with a newly good fund in an n-year lockup. Otherwise, the fund becomes a 2-year-old sick fund ($S2$) or a 2-year-old good fund ($G2$) with the probability q and r, respectively.

In the transition matrix P_3 in (9.1), we have no dead states, because we intend to use persistence to calculate the expected returns looking at the beginning of a period after transition out of a dead state.

Similarly, the transition matrix for an n-year lockup in the rolling lockup model is:

$$P_n = \begin{matrix} G0 \\ G1 \\ G2 \\ \dots \\ G(n-1) \\ S1 \\ S2 \\ \dots \\ S(n-1) \end{matrix} \begin{pmatrix} 0 & p & 0 & 0 & \dots & 0 & 1-p & 0 & 0 & \dots & 0 \\ 0 & 0 & p & 0 & \dots & 0 & 0 & 1-p & 0 & \dots & 0 \\ & & & & \dots & & & & & & \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 1-q-r & 0 & q & 0 & \dots & 0 & 0 & r & 0 & \dots & 0 \\ 1-q-r & 0 & 0 & q & \dots & 0 & 0 & 0 & r & \dots & 0 \\ & & & & \dots & & & & \dots & & \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{9.2}$$

We compute the steady state probability vector for an n-year lockup, π_n, from the following equation:

$$\pi_n = \pi_n \cdot P_n. \tag{9.3}$$

We then consider the steady-state probabilities of a good and sick state, respectively:

$$\pi_{n,G} = \pi_{n,G0} + \pi_{n,G1} + \dots + \pi_{n,G(n-1)} \tag{9.4}$$

and

$$\pi_{n,S} = \pi_{n,S1} + \pi_{n,S2} + ... + \pi_{n,S(n-1)} \tag{9.5}$$

The expected return of the n-year rolling lockup fund at the steady state is expressed as

$$R_n = \pi_{n,G} \cdot \gamma_G \cdot Y_G + \pi_{n,S} \cdot \gamma_S \cdot Y_S \ . \tag{9.6}$$

To explain (9.6), we are using the persistence factors (γ_G, γ_S) to avoid explicitly considering the death. Given that a fund is in a good state at the beginning of the period, the expected return at the end of the period is expressed as $\gamma_G \cdot Y_G$. Similarly, we use $\gamma_S \cdot Y_S$ if a fund is in a sick state at the beginning of the period.

Using the parameter values in the DTMC model ($\gamma_G = \gamma_S = 0.5, \sigma = 0.1$), we calculate R_n with $\delta = 0, 0.03$, and 0.06. Figure 26 shows the lockup premium in the rolling lockup model.

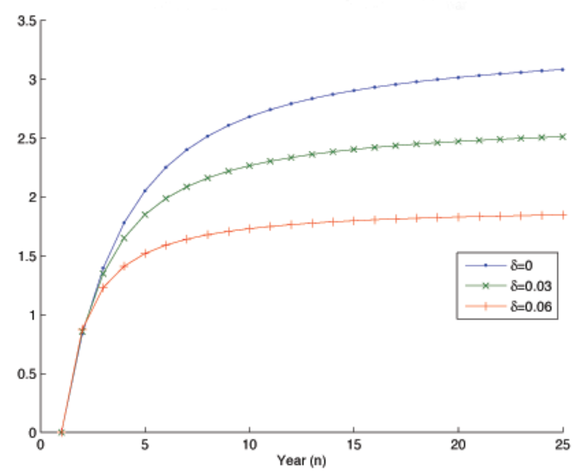


Figure 26: The lockup premium (percentage points of return) as a function of n in the rolling lockup model for $\delta = 0, 0.03$, and 0.06 with the parameter values obtained in the DTMC model.