Modeling Equity Market Behavior

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Organization of the talk

1. Introduction & References
2. The logarithmic representation
3. Markets with constant parameters
4. Volatility-stabilized markets
5. Rank-based models

Introduction

It is possible to construct relatively simple stochastic stock market models that exhibit selected characteristics of real stock markets, so that an understanding of these models will provide some insight into the behavior of actual markets.

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The logarithmic representation

Consider a market $\mathcal{M}$ consisting of n stocks represented by their *capitalization processes* $X_1, \ldots, X_n$, such that X_i evolves according to

$$d \log X_i(t) = \gamma_i(t)\, dt + \sum_{\nu=1}^{d} \xi_{i\nu}(t)\, dW_\nu(t), \tag{1}$$

where $(W_1, \ldots, W_d)$ is d-dimensional Brownian motion with $d \geq n$, and the γ_i and $\xi_{i\nu}$ are measurable, adapted, and satisfy certain regularity conditions. For well-behaved $\xi_{i\nu}$,

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_i(T) - \int_0^T \gamma_i(t)\, dt \Big) = 0 \quad \text{a.s.}$$

The process γ_i is called the *growth rate process* for X_i, and the *covariance process* for X_i and X_j is given by

$$\sigma_{ij}(t) \triangleq \frac{d}{dt}\langle \log X_i, \log X_j \rangle_t = \sum_{\nu=1}^{d} \xi_{i\nu}(t)\xi_{j\nu}(t).$$

We shall assume that $\sigma(t) = \big(\sigma_{ij}(t)\big)$ is nonsingular.

A *portfolio* $\pi = (\pi_1, \ldots, \pi_n)$, is a measurable, adapted, and bounded process with $\pi_1(t) + \cdots + \pi_n(t) = 1$. If we let $Z_\pi(t) > 0$ represent the value of π at time t, then

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)}.$$

By Itô's rule, in logarithmic form this becomes

$$d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\, d \log X_i(t) + \gamma_\pi^*(t)\, dt,$$

with *excess growth rate*

$$\gamma_\pi^*(t) \triangleq \frac{1}{2}\Big(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t) \Big).$$

The portfolio growth rate

$$\gamma_\pi(t) \triangleq \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t)$$

determines the long-term behavior of Z_π, as for stocks.

Let η be a portfolio with relative covariances

$$\tau_{ij}^{\eta}(t) \triangleq \frac{d}{dt}\langle \log(X_i/Z_\eta), \log(X_j/Z_\eta)\rangle_t.$$

Then γ_π^* is *numeraire invariant*, with

$$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\tau_{ij}^{\eta}(t)\Big),$$

and, in particular,

$$\gamma_\pi^*(t) = \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\pi}(t).$$

This will be non-negative for a long-only portfolio.

The market portfolio

For the market $\mathcal{M}$ let

$$X(t) \triangleq X_1(t) + \cdots + X_n(t) \quad \text{and} \quad \mu_i(t) \triangleq \frac{X_i(t)}{X(t)}.$$

Then μ is called the *market portfolio* and the μ_i are the *market weights*. Let $Z_\mu(0) = X(0)$, and then

$$\frac{dZ_\mu(t)}{Z_\mu(t)} = \sum_{i=1}^{n} \mu_i(t) \frac{dX_i(t)}{X_i(t)} = \sum_{i=1}^{n} \frac{dX_i(t)}{X(t)} = \frac{dX(t)}{X(t)}.$$

Let $\mu_{(1)}(t) \geq \cdots \geq \mu_{(n)}(t)$ be the ranked market weights.

$\mathcal{M}$ is *coherent* if $\lim\limits_{t\to\infty} t^{-1} \log \mu_{(n)}(t) = 0$.

$\mathcal{M}$ is *diverse* if $\mu_{(1)}(t) \ll 1$.

Let $\varepsilon(t)$ be the minimum eigenvalue of $\sigma(t)$. $\mathcal{M}$ is *uniformly nondegenerate* if $\varepsilon(t) \gg 0$.

$\mathcal{M}$ admits *strong relative arbitrage* if there exist π, η and $T > 0$ such that $Z_\pi(0) = Z_\eta(0)$ and $P\{Z_\pi(T) > Z_\eta(T)\} = 1$. It is *weak arbitrage* if $P\{Z_\pi(T) \geq Z_\eta(T)\} = 1$ and $P\{Z_\pi(T) > Z_\eta(T)\} > 0$.

Lemma. Let $\varepsilon(t)$ be the minimum eigenvalue of $\sigma(t)$. Then

$$\gamma_\mu^*(t) \geq \frac{\varepsilon(t)}{2}\big(1 - \mu_{(1)}(t)\big)^2.$$

Proof. Let e_i be the ith unit vector in $\mathbb{R}^n$. Then

$$\begin{aligned} \tau_{ii}^\mu(t) &= \big(e_i - \mu(t)\big)\sigma(t)\big(e_i - \mu(t)\big)' \geq \varepsilon(t)\big\|e_i - \mu(t)\big\|^2 \\ &\geq \varepsilon(t)\big(1 - \mu_i(t)\big)^2 \geq \varepsilon(t)\big(1 - \mu_{(1)}(t)\big)^2. \end{aligned}$$

Hence,

$$\gamma_\mu^*(t) = \frac{1}{2}\sum_{i=1}^n \mu_i(t)\tau_{ii}^\mu(t) \geq \frac{\varepsilon(t)}{2}\big(1 - \mu_{(1)}(t)\big)^2. \qquad \square$$

Markets with constant parameters

Proposition. In a market with constant parameters,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu_{(1)}(t)\, dt = 1.$$

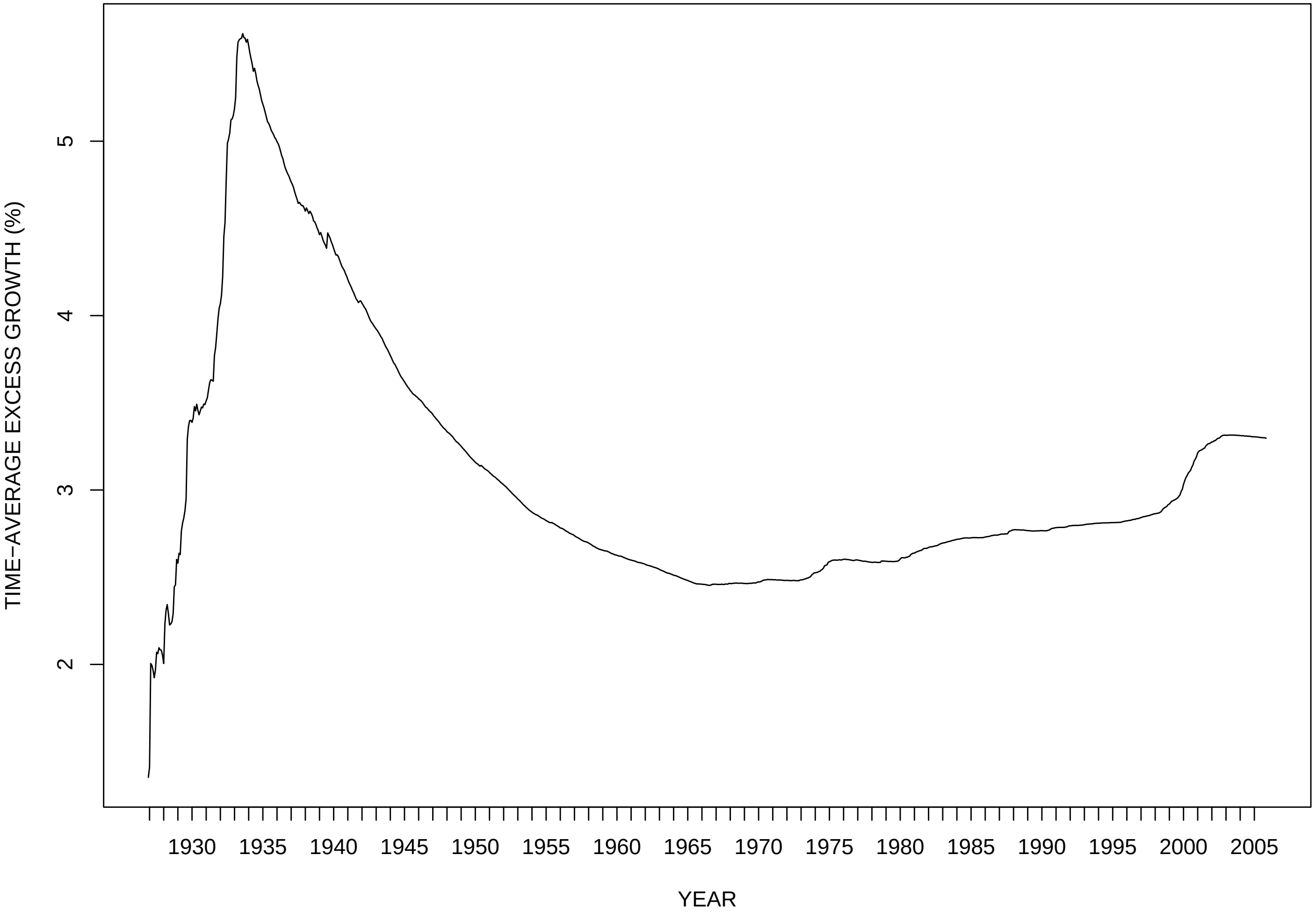
Proof. We can ignore all the stocks except those that share the highest growth rate, γ. Then

$$\lim_{T\to\infty} \frac{1}{T} \log Z_\mu(T) = \gamma + \lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_\mu^*(t)\, dt,$$

and since Z_μ cannot grow faster than all the stocks, the last term must vanish. But $\varepsilon(t) = \varepsilon(0) > 0$ is constant, so

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_\mu^*(t)\, dt &\geq \frac{\varepsilon(0)}{2} \lim_{T\to\infty} \frac{1}{T} \int_0^T \big(1 - \mu_{(1)}(t)\big)^2\, dt \\ &\geq \frac{\varepsilon(0)}{2} \Big(\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(1 - \mu_{(1)}(t)\big)\, dt \Big)^2. \quad \square\end{aligned}$$

U.S. market, 1927–2005



$$\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt.$$

Proposition. $\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \gg 0$ implies strong relative arbitrage.

Proof. Suppose that $\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt > \delta > 0$. Let $c > 0$ and $\mathbf{S}_c(\mu(t)) \triangleq c - \sum_{i=1}^n \mu_i(t)\log\mu_i(t)$, so $c \leq \mathbf{S}_c(\mu(t)) \leq c + \log n$. Let π be the portfolio with $\pi_i(t) = \frac{c - \mu_i(t)\log\mu_i(t)}{\mathbf{S}_c(\mu(t))}$. Then

$$\log\left(\frac{Z_\pi(T)/Z_\pi(0)}{Z_\mu(T)/Z_\mu(0)}\right) = \log\frac{\mathbf{S}_c(\mu(T))}{\mathbf{S}_c(\mu(0))} + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{S}_c(\mu(t))}dt,$$

and we have arbitrage for $T \geq \log(1 + c^{-1}\log n)(c + \log n)/\delta$. □

Note. D. Fernholz & I. Karatzas (2008) have proved short-term strong relative arbitrage in uniformly nondegenerate Markovian markets with $\gamma_\mu^*(t) \gg 0$.

A volatility-stabilized market

Consider the market

$$d \log X_i(t) = \frac{dW_i(t)}{\sqrt{\mu_i(t)}}, \quad i = 1, \ldots, n.$$

A weak solution exists and is unique in distribution (Bass & Perkins (2002)). Here

$$\gamma_\mu(t) = \gamma_\mu^*(t) = \frac{1}{2}\Big(\sum_{i=1}^{n} \mu_i(t)\sigma_{ii}(t) - \sum_{i=1}^{n} \mu_i^2(t)\sigma_{ii}(t) \Big) = \frac{n-1}{2},$$

and $\sigma_{\mu\mu}(t) = 1$, so Z_μ is a Brownian motion plus a constant drift. $\mathcal{M}$ is not diverse, but admits long-term strong relative arbitrage since $\gamma_\mu^*(t) = (n-1)/2 > 0$. Short-term weak relative arbitrage also holds.

Note. A. Banner & D. Fernholz (2007) showed that short-term strong relative arbitrage exists in this market.

Although Z_μ is relatively well-behaved, the individual X_i go all over the place:

$$\lim_{t\to\infty} \frac{1}{t}\log X_i(t) = \frac{n-1}{2} \quad \text{in probability, while}$$

$$\limsup_{t\to\infty} \frac{1}{t}\log X_i(t) = \frac{n-1}{2}, \quad \liminf_{t\to\infty} \frac{1}{t}\log X_i(t) = -\infty \quad \text{a.s.}$$

We see that the market is not coherent, and the γ_i do not determine the long-term behavior of the stocks.

The market behavior is similar for the generalized model

$$d\log X_i(t) = \frac{\alpha\, dt}{2\mu_i(t)} + \frac{dW_i(t)}{\sqrt{\mu_i(t)}}, \quad i = 1, \dots, n,$$

with $\alpha > 0$, but for $\alpha < 0$ the behavior is quite pathological.

Rank-based models

Let us assume that meas$\{t \mid X_i(t) = X_j(t)\} = 0$ for $i \neq j$, a.s., and that there are no triple points, i.e., for $i < j < k$,

$$\{t \mid X_i(t) = X_j(t) = X_k(t)\} = \varnothing, \quad \text{a.s.}$$

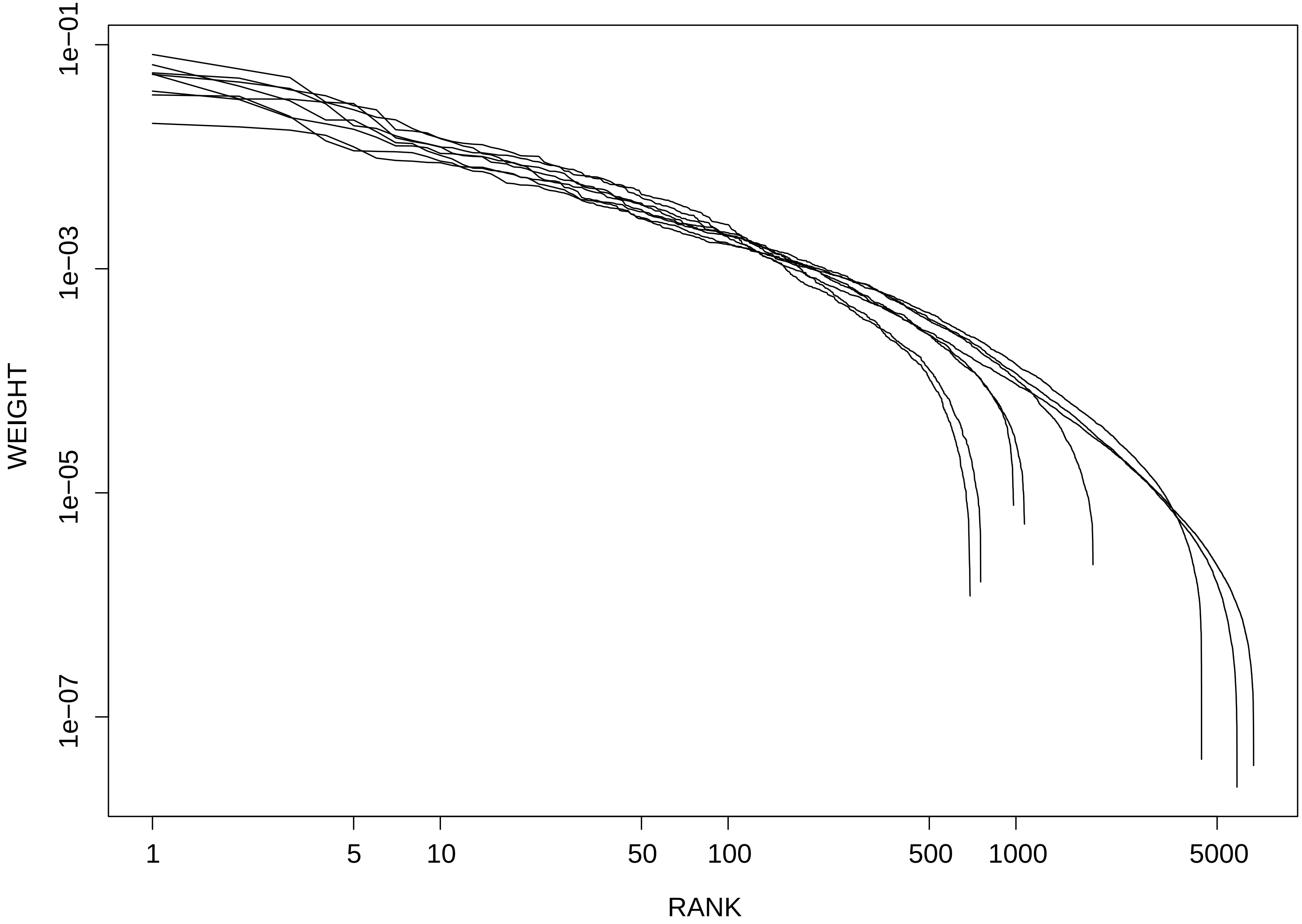
Consider the ranked market capitalizations $X_{(1)}(t) \geq \cdots \geq X_{(n)}(t)$ and the ranked market weights $\mu_{(1)}(t) \geq \cdots \geq \mu_{(n)}(t)$. Let p_t be the random permutation of $\{1, \ldots, n\}$ such that for $k = 1, \ldots, n$,

$$\begin{aligned} X_{p_t(k)}(t) &= X_{(k)}(t), \\ p_t(k) < p_t(k+1) \quad &\text{if} \quad X_{(k)}(t) = X_{(k+1)}(t). \end{aligned}$$

Define r_t to be the inverse of p_t, so $r_t(i)$ is the rank of $X_i(t)$.

We call the log-log plot of the ranked market weights the *capital distribution curve.*

Capital distribution curves: 1929–1999.



Definition. The *local time* (at 0) for X is defined by

$$\Lambda_X(t) \triangleq \frac{1}{2}\Big(|X(t)| - |X(0)| - \int_0^t \operatorname{sgn}(X(s))\, dX(s)\Big),$$

where
$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x \leq 0 \end{cases}$$

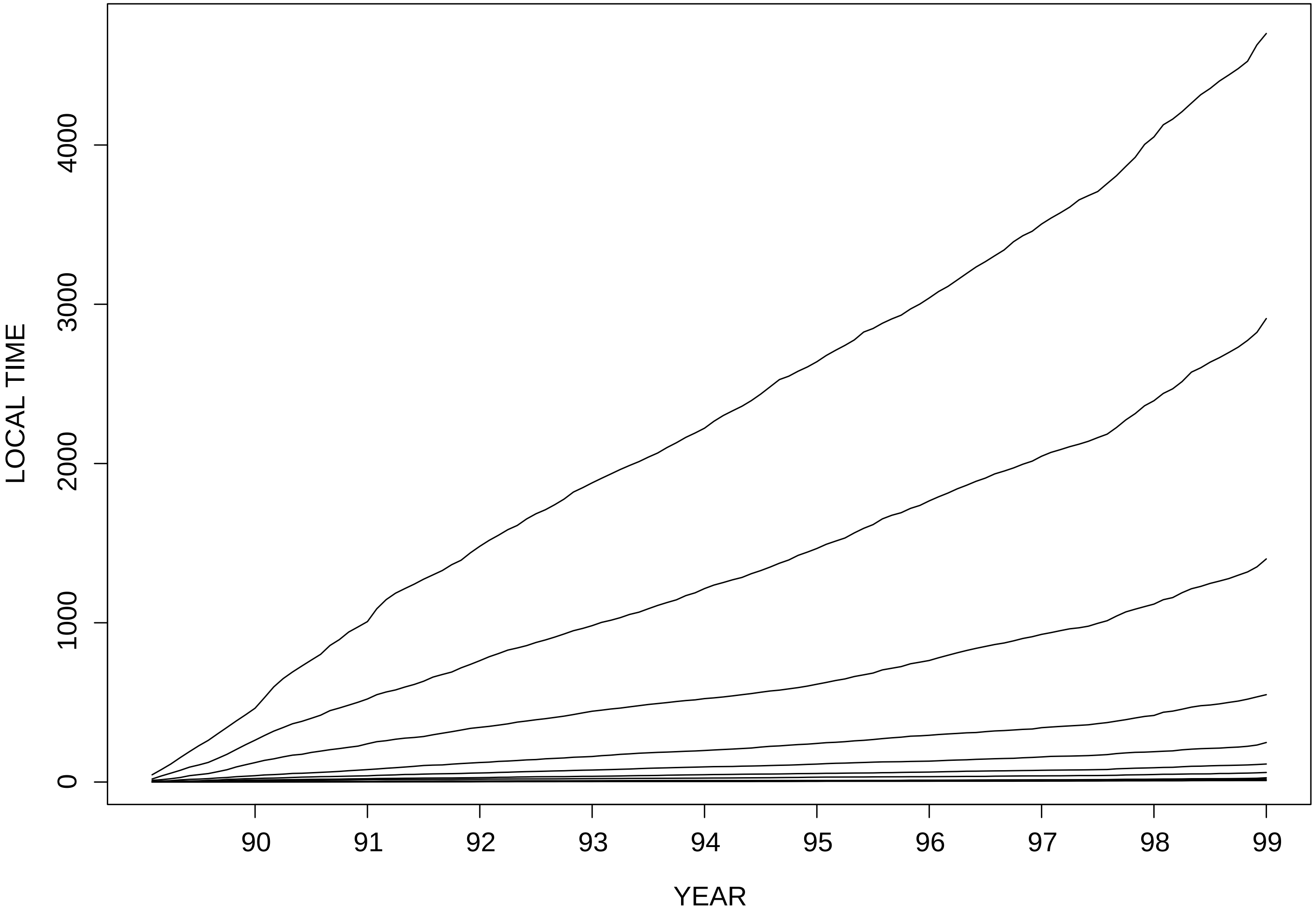
(Tanaka-Meyer).

Proposition. Let us use the notation $\Lambda_{k,k+1} \triangleq \Lambda_{\log \mu_{(k)} - \log \mu_{(k+1)}}$. Then

$$\log \mu_{(k)}(t) = \int_0^t d \log \mu_{p_s(k)}(s) + \frac{1}{2}\Lambda_{k,k+1}(t) - \frac{1}{2}\Lambda_{k-1,k}(t).$$

Note. A. Banner & R. Ghomrasni (2007) have generalized this to include multiple points.

U.S. market, 1990–1999



$$\Lambda_{k,k+1}(t),\ k = 10, 20, 40, \ldots, 5120.$$

Definition. The market $\mathcal{M}$ is *asymptotically stable* if it is coherent and, for $k = 1, \ldots, n-1$, a.s.:

(*i*) $\boldsymbol{\lambda}_{k,k+1} \triangleq \lim_{t\to\infty} t^{-1}\Lambda_{k,k+1}(t) > 0;$

(*ii*) $\boldsymbol{\sigma}^2_{k:k+1} \triangleq \lim_{t\to\infty} t^{-1}\langle \log\mu_{(k)} - \log\mu_{(k+1)}\rangle_t > 0.$

Proposition. In an asymptotically stable market,

$$\boldsymbol{g}_k \triangleq \lim_{t\to\infty} \frac{1}{t}\int_0^t \big(\gamma_{p_s(k)}(s) - \gamma_\mu(s)\big)\,ds = \frac{1}{2}\boldsymbol{\lambda}_{k-1,k} - \frac{1}{2}\boldsymbol{\lambda}_{k,k+1}.$$

Proof.

$$\log\mu_{(k)}(t) = \int_0^t \big(\gamma_{p_s(k)}(s) - \gamma_\mu(s)\big)\,ds + \frac{1}{2}\Lambda_{k,k+1}(t) - \frac{1}{2}\Lambda_{k-1,k}(t) + M_k(t),$$

where M_k is a local martingale with $\langle M_k\rangle_t = O(t)$.

We have $\log \mu_{(k)}(t) \cong \boldsymbol{g}_k t + \frac{1}{2}\Lambda_{k,k+1}(t) - \frac{1}{2}\Lambda_{k-1,k}(t) + M_k(t)$, so

$$\begin{aligned}\log \mu_{(k)}(t) - \log \mu_{(k+1)}(t) &\cong (\boldsymbol{g}_k - \boldsymbol{g}_{k+1})t - \frac{1}{2}\Lambda_{k-1,k}(t) \\ &\quad - \frac{1}{2}\Lambda_{k+1,k+2}(t) + \Lambda_{k,k+1}(t) + M(t), \\ &\cong -\boldsymbol{\lambda}_{k,k+1}\, dt + \Lambda_{k,k+1}(t) + \boldsymbol{\sigma}^2_{k:k+1} B(t),\end{aligned}$$

where $\langle M \rangle_t = \boldsymbol{\sigma}^2_{k:k+1} t$ and B is Brownian motion. Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\log \mu_{(k)}(t) - \log \mu_{(k+1)}(t)\big) dt = \frac{\boldsymbol{\sigma}^2_{k:k+1}}{2\boldsymbol{\lambda}_{k,k+1}},$$

for $k = 1, \ldots, n-1$, so, for large enough k,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\log \mu_{(k)}(t) - \log \mu_{(k+1)}(t)}{\log(k) - \log(k+1)}\, dt \cong -\frac{k\boldsymbol{\sigma}^2_{k:k+1}}{2\boldsymbol{\lambda}_{k,k+1}}.$$

The "Atlas" model

Consider the market $X_1, \ldots, X_n$, with

$$d \log X_i(t) = \gamma_i(t)\, dt + \sigma\, dW_i(t),$$

where
$$\gamma_i(t) = \begin{cases} ng & \text{if } X_i(t) = X_{(n)}(t), \\ 0 & \text{otherwise.} \end{cases}$$

A weak solution exists and is unique in distribution. An Atlas model admits an equivalent probability measure under which it is Brownian motion, so $\text{meas}\{t \mid X_i(t) = X_j(t)\} = 0$ for $i \neq j$, a.s., and there are no triple points (and no arbitrage). T. Ichiba (2007) and S. Pal & J. Pitman (2007) have shown that the random variables

$$\Xi_k \triangleq \lim_{t\to\infty} \big(\log \mu_{(k)}(t) - \log \mu_{(k+1)}(t) \big) \quad \text{in distribution}$$

are independent and exponentially distributed, so the heuristics for asymptotically stable markets are valid for Atlas models.

We have,

$$\lim_{t\to\infty} \frac{1}{t}\big(\log X_1(t) + \cdots + \log X_n(t)\big) = ng,$$

and it can be shown that for all i and k,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_{\{X_i(t)=X_{(k)}(t)\}}\,dt = \frac{1}{n}, \quad \text{so} \quad \lim_{t\to\infty} \frac{1}{t}\log X_i(t) = g.$$

If the stocks all grow at rate g, then so must the market, so

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \gamma_\mu(t)\,dt = g,$$

and therefore,

$$\boldsymbol{g}_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_{p_t(k)}(t) - \gamma_\mu(t)\big)\,dt = \begin{cases} -g & \text{for } k < n, \\ (n-1)g & \text{for } k = n. \end{cases}$$

Hence,

$$\frac{1}{2}\boldsymbol{\lambda}_{k-1,k} - \frac{1}{2}\boldsymbol{\lambda}_{k,k+1} = \boldsymbol{g}_k = -g$$

for $k = 1, \dots, n-1$, so

$$\begin{aligned}\boldsymbol{\lambda}_{k,k+1} &= -2\big(\boldsymbol{g}_1 + \cdots + \boldsymbol{g}_k\big) \\ &= 2kg.\end{aligned}$$

We have $\boldsymbol{\sigma}^2_{k:k+1} = 2\sigma$, so the slope of the capital distribution curve will be

$$-\frac{k\boldsymbol{\sigma}^2_{k:k+1}}{2\boldsymbol{\lambda}_{k,k+1}} = -\frac{\sigma^2}{2g},$$

and the capital distribution is Pareto, with

$$\mu_{(k)} \propto k^{-\sigma^2/2g}.$$

Portfolio optimization in open markets

Let $\widehat{\mu}$ represent the an Atlas model in which the smallest stock is "off limits". We call this a *truncated market*, with stocks $(X_{(1)}, \ldots, X_{(n-1)})$. This is an example of an *open market*, and we are interested in portfolio behavior in these markets. Let us consider η and $\widehat{\eta}$, equal-weighted portfolios in μ and $\widehat{\mu}$, respectively. Since in $\widehat{\mu}$ all the stocks have the same parameters, $\widehat{\eta}$ is likely to be optimal in some sense.

It is convenient to use boldface to denote time-average values for large n, so, for example,

$$\boldsymbol{\gamma}_\mu \triangleq \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_\mu(t)\, dt = g.$$

For $\sigma^2/2g > 1$, we have $\mu_{(n)} ng = O(n^{1-\sigma^2/2g}) \to 0$, so

$$\boldsymbol{\gamma}_{\widehat{\mu}} = \boldsymbol{\gamma}^*_{\widehat{\mu}} = \boldsymbol{\gamma}^*_{\mu} = \boldsymbol{\gamma}_{\mu} = g, \quad \text{and} \quad \boldsymbol{\gamma}_{\widehat{\eta}} = \boldsymbol{\gamma}^*_{\widehat{\eta}} = \frac{\sigma^2}{2} > g.$$

For $\sigma^2/2g \leq 1$,

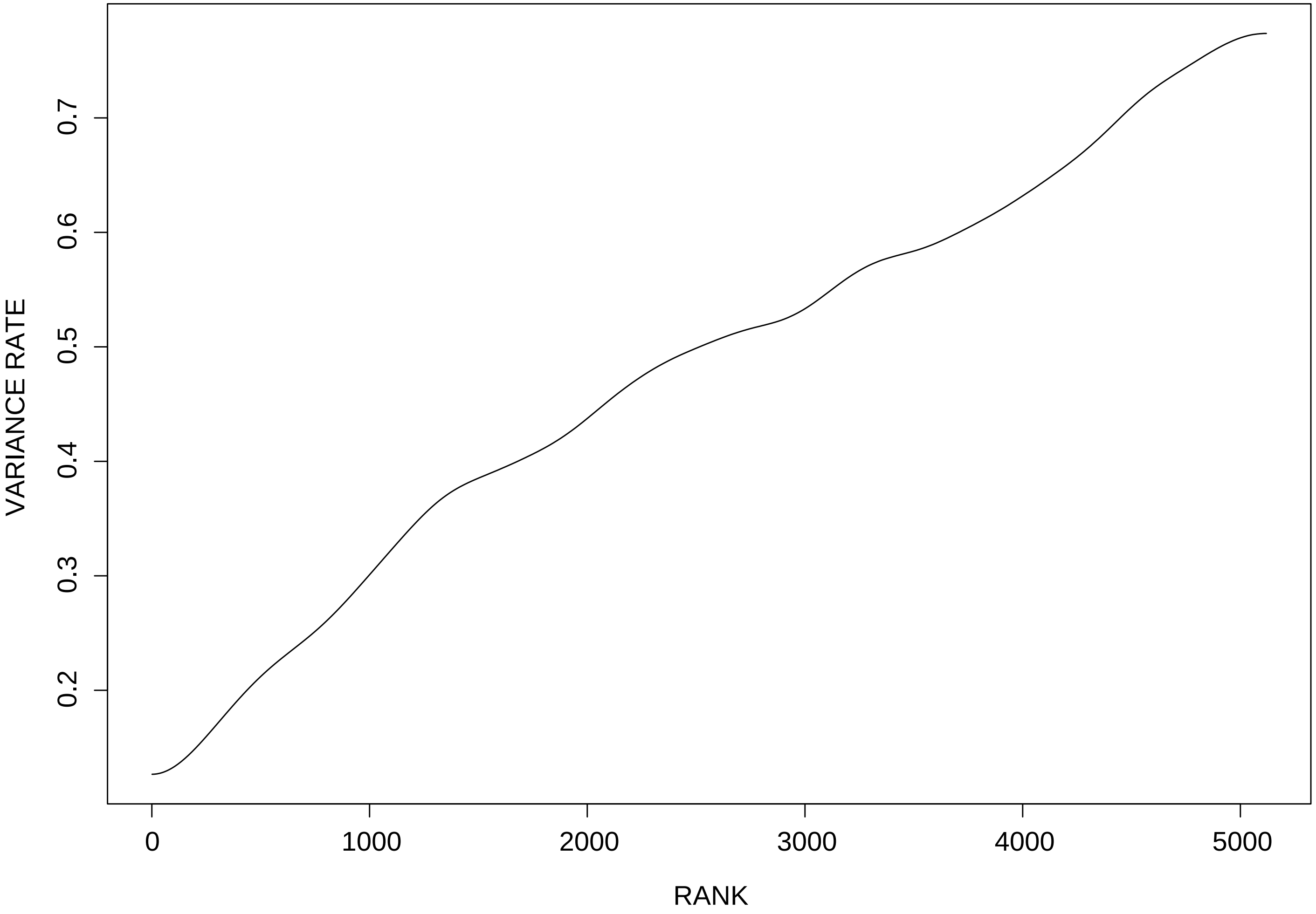
$$\boldsymbol{\gamma}^*_{\mu} = \frac{\sigma^2}{2}\Big(1 - \lim_{n\to\infty} \sum_{k=1}^{n} \mu^2_{(k)}\Big) = \frac{\sigma^2}{2}\big(1 - O(\mu_{(1)})\big) = \frac{\sigma^2}{2} \leq g = \gamma_{\mu}.$$

From this, we have

$$\boldsymbol{\gamma}_{\widehat{\mu}} = \boldsymbol{\gamma}^*_{\widehat{\mu}} = \boldsymbol{\gamma}^*_{\mu} = \frac{\sigma^2}{2}, \quad \text{and} \quad \boldsymbol{\gamma}_{\widehat{\eta}} = \boldsymbol{\gamma}^*_{\widehat{\eta}} = \boldsymbol{\gamma}^*_{\eta} = \frac{\sigma^2}{2}.$$

This indicates that for large n and $\sigma^2/2g \leq 1$, truncated Atlas models may be examples of (almost) "efficient markets".

U.S. market, 1990–1999



Smoothed annualized values of $\boldsymbol{\sigma}^2_{k:k+1},\ k = 1, \ldots, 5119.$

First-order models

Suppose we have constants $g_1, \ldots, g_n$ with $g_1 + \cdots + g_n = 0$, and $\sigma_1 > 0, \ldots, \sigma_n > 0$, such that

$$(*) \qquad \sum_{k=1}^{m} g_k < 0, \quad m < n.$$

Then the market $X_1, \ldots, X_n$ with

$$d \log X_i(t) = g_{r_t(i)}\, dt + \sigma_{r_t(i)}\, dW_i(t),$$

is called a *first-order model*. Condition $(*)$ is needed for asymptotic stability. There can be a problem with triple points. T. Ichiba has shown that if

$$\max_{1 \le k \le n} \sigma_k^2 < \frac{1}{2} \sum_{k=1}^{n} \sigma_k^2, \quad \text{and} \quad 0 \le \sigma_k^2 - \sigma_{k-1}^2 \le \sigma_{k+1}^2 - \sigma_k^2,$$

$1 < k < n$, then there will be no triple points.

Suppose $\boldsymbol{g}_k$ and $\boldsymbol{\sigma}^2_{k:k+1}$ are the parameters estimated from the U.S. market, and consider a first-order model with

$$g_k = \boldsymbol{g}_k, \qquad \text{and} \qquad \sigma_k^2 = \frac{1}{4}(\boldsymbol{\sigma}^2_{k-1:k} + \boldsymbol{\sigma}^2_{k:k+1}),$$

with $\qquad \sigma_1^2 = \frac{1}{2}(\boldsymbol{\sigma}^2_{1:2}) \qquad \text{and} \qquad \sigma_n^2 = \frac{1}{2}(\boldsymbol{\sigma}^2_{n-1:n}).$

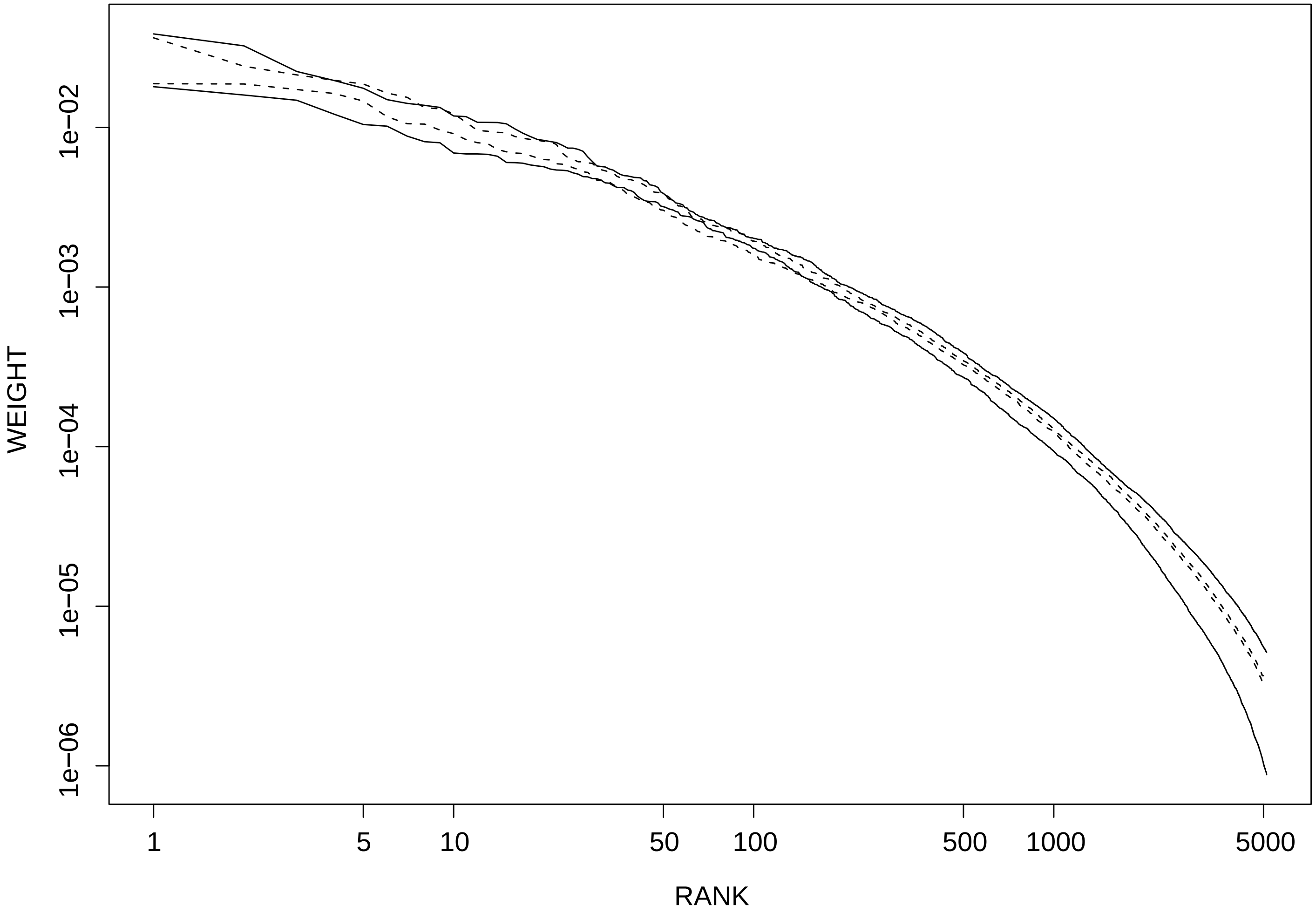
The asymptotic parameters for this model will be

$$\overline{\boldsymbol{g}}_k = \boldsymbol{g}_k,$$

$$\overline{\boldsymbol{\sigma}}^2_{k:k+1} = \frac{1}{4}\boldsymbol{\sigma}^2_{k-1:k} + \frac{1}{2}\boldsymbol{\sigma}^2_{k:k+1} + \frac{1}{4}\boldsymbol{\sigma}^2_{k+1:k+2},$$

so the capital distribution curves should be similar.

Envelopes of U.S. capital distribution curves.



1990–1999, solid lines; simulated data, broken lines.

Hybrid models

Suppose we have constants $g_1, \ldots, g_n$ with $g_1 + \cdots + g_n = 0$, $\gamma_1 \geq \cdots \geq \gamma_n$ with $\gamma_1 + \cdots + \gamma_n = 0$, and $\sigma_1 > 0, \ldots, \sigma_n > 0$, $\rho_1 \geq 0, \ldots, \rho_n \geq 0$, such that

$$(*) \qquad \sum_{k=1}^{m} g_k + \gamma_k < 0, \quad m < n.$$

Then the market $X_1, \ldots, X_n$ with

$$d \log X_i(t) = \big(g_{r_t(i)} + \gamma_i\big)\, dt + \sigma_{r_t(i)}\, dW_i(t) + \rho_i\, dB_i(t),$$

where $W = (W_1, \ldots, W_n)$ and $B = (B_1, \ldots, B_n)$ are independent Brownian motions, is called a *hybrid model*. Condition $(*)$ is needed for asymptotic stability, and conditions are needed to avoid triple points.

I. Karatzas has shown that for $i, k = 1, \ldots, n$,

$$\theta_{ki} \triangleq \lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{X_i(t) = X_{(k)}(t)\}}\, dt$$

satisfy

$$\sum_{k=1}^{n} \theta_{ki}\big(g_k + \gamma_i\big) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \big(g_{r_t(i)} + \gamma_i\big)\, dt = 0,$$

$$\sum_{i=1}^{n} \theta_{ki}\big(g_k + \gamma_i\big) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \big(g_k + \gamma_{p_t(k)}\big)\, dt = \boldsymbol{g}_k.$$

A. Banner has shown that for

$$\Theta^{\pi} \triangleq \prod_{k=1}^{n-1} \frac{1}{-\big(g_1 + \cdots + g_k + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(k)}\big)},$$

where $\pi \in \Sigma^n$, and

$$\Theta_{ki} \triangleq \sum_{\pi:\pi(k)=i} \Theta^{\pi},$$

then

$$\sum_{k=1}^{n} \Theta_{ki}\big(g_k + \gamma_i\big) = 0.$$

T. Ichiba has shown that if $\sigma_k^2 = s^2 + k\sigma^2$, and $\rho_i = 0$, then the θ_{ki} will be proportional to the Θ_{ki}. For general σ_k and ρ_i, this is no longer true.