First Passage Times and Analytical Solutions for Options on Two Assets with Jump Risk

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Abstract

Payoffs of many options involve two or more assets. The paper gives analytical solutions for two-dimensional barrier options under a jump diffusion model with jump sizes having a mixture of multivariate asymmetric Laplace distributions. The analytical solutions are made possible because we find the distribution of related two-dimensional first passage times. The model is rich enough to incorporate both correlated common jumps and individual jumps. Analytical formulae for exchange options are also given. Numerical examples indicate that the formulae are easy to implement and are accurate.
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1 Introduction

Many options traded in exchanges and in over-the-counter markets, such as two-dimensional barrier options and exchange options, have payoffs depending on more than one assets. This paper gives analytical solutions for these options with jump risk, by first solving a related first-passage-time problem.

A two-dimensional barrier option has a regular call or put payoff from one asset while the barrier crossing is determined by another asset. For example, in late 1993 Bankers Trust issued a call option on a basket of Belgian stocks which would be knocked out if the Belgian franc appreciated by more than 30% (Zhang, 1998); in this case we have a up-and-out call option. There are eight types of two-dimensional barrier options: up (down)-and-in (out) call (put) options. Mathematically, the payoff of a two-dimensional up-an-in put barrier option is $(K - S_1(T))^+ 1_{\{\max_{0\le t\le T} S_2(t)\ge H\}}$, where $S_i(t), i = 1, 2$, are prices of two assets, $K > 0$ is the strike price of the put option and H is the barrier level. To price this option, it is crucial to compute the joint distribution of the first passage time

$$\mathsf{P}(X_T^{(1)} \leq a,\ \max_{0\leq s\leq T} X_s^{(2)} \geq b) = \mathsf{P}(X_T^{(1)} \leq a,\ \tau_b \leq T),$$

where the first passage time τ_b is defined to be $\tau_b \equiv \tau_b^{(2)} := \inf\{t \geq 0 : X_t^{(2)} \geq b\}$, $b > 0$. Here $X_T^{(i)} = \log(S_i(T)/S_i(0))$ is the return process for the ith asset, $i = 1, 2$. An exchange option gives the holder the right to exchange one asset to another asset. More precisely, the payoff of an exchange option is $(S_1(T) - e^{-k}S_2(T))^+$, where e^{-k} is the ratio of the shares to be exchanged.

Analytical solutions for these options are available under the classical Brownian models; see, e.g., the books by Hull (2006) and Zhang (1998). One of the well-documented empirical features in asset pricing is the presence of jumps in return processes; see Kijima (2002) and Cont and Tankov (2004). However, it becomes difficult to retain analytical tractability after jumps being introduced. This is partly because of the "overshoot" problem, due to the possibility of jumping over the barrier. One way to get around the overshoot problem is to make the jump size distribution to be two-sided exponential, as the exponential distribution has the memoryless property. This is what is achieved in Kou and Wang (2004), and Kou, Petrella, and Wang (2005) for one-dimensional options.

The current paper extends the previous one-dimensional results by giving analytical solutions for two-dimensional barrier options and exchange options. The contribution is threefold: (1) In terms of modelling, we propose a multivariate jump diffusion model with both correlated common jumps and individual jumps. The jump sizes have a multivariate asymmetric Laplace distribution (which is related but not equal to the double exponential distribution). The model not only provides a flexible framework to study correlated jumps but also is amenable for computation, especially for barrier options. (2) In terms of mathematics, we provide analytical solutions to the first passage time problem in two dimension. Compared to the one-dimensional case the two-dimensional problem poses some technical challenges. First, with both common jumps and individual jumps, the generator of the two-dimensional process becomes more involved. Second, because the joint density of the asymmetric Laplace distribution has no analytical expression, the calculation related to the joint density and generator becomes complicated (see Appendix B). Third, one has to use several uniform integrability arguments to substantiate a martingale argument, as Itô's formula cannot be applied directly due to discontinuity (see the proof in Theorem 3.2). (3) In terms of financial engineering, we give analytical solutions for barrier and exchange options and other related options.

In terms of the related literature, Cont and Voltchkova (2005) provided a numerical method (based on the finite difference method) to price one-dimensional path-dependent options with jumps. Davydov and Linetsky (2001) provided analytical formulae for barrier options under the CEV model (without jumps). Feng and Linetsky (2005) and Feng, Linetsky and Marcozzi (2004) showed how to price path-dependent options numerically via variational methods and extrapolation. The emphasis of the current paper is on analytical solutions and Laplace transforms for two-dimensional options, which is different from the existing literature.

More general models will have both stochastic volatility and jumps, especially for long term options. Therefore, the formulae given in this paper is only a first step to get analytical solutions for options with multivariate payoffs under more general models. However, the analytical formulae presented here can be useful for short-term options, and can provide a useful benchmark for more complicated models (for which one perhaps has to resort to simulation and other numerical procedures).

The remainder of the paper is organized as follows. After the model is introduced in Section 2, we study the first passage time problems in Section 3. Barrier and exchange options are discussed in Sections 4 and 5, respectively. Technical remarks and proofs are given in four Appendices.

2 The Model

2.1 Asymmetric Laplace Distribution

The common jumps in the multivariate jump diffusion model to be introduced next will have a multivariate asymmetric Laplace distribution. An n-dimensional asymmetric Laplace random vector Y, denoted by $Y \sim \mathcal{AL}_n(m, J)$, is defined via its characteristic function

$$\Psi_Y(\theta) = \mathsf{E}[e^{i\theta' Y}] = \frac{1}{1 + \frac{1}{2}\theta' J \theta - i m' \theta}, \tag{1}$$

where $m \in R^n$ and J is a $n \times n$ positive definite symmetric matrix. The requirement of the matrix J being positive definite is postulated to guarantee that the n-dimensional distribution is non-degenerate; otherwise, the dimension of the distribution may be less than n. The vector m is the mean $\mathsf{E}[Y] = m$ and the matrix J plays a role similar to that of the variance and covariance matrix. In the case of the univariate Laplace distribution, the characteristic function in (1) becomes

$$\Psi_Y(\theta) = \frac{1}{1 + \frac{1}{2}v^2\theta^2 - i m \theta}, \tag{2}$$

where v^2 is the equivalence of J in (1). For further information about the asymmetric Laplace distribution, see Kotz et al. (2001).

The asymmetric Laplace distribution has many properties similar to those of the multivariate normal distribution. This can be easily seen from the fact that

$$Y \stackrel{d}{=} mB + B^{1/2} Z, \tag{3}$$

where $Z \sim N_n(0, J)$ is a multivariate normal distribution with mean 0 and covariance matrix J, and B is a one-dimensional exponential random variable with mean 1, independent of Z. For example, for the kth component of Y we have $Y^{(k)} \stackrel{d}{=} m_k B + B^{1/2} Z_k$ with $B \sim \exp(1)$ and $Z_k \sim N(0, J_{kk})$, which implies that the marginal distribution of $Y^{(k)}$ has a univariate asymmetric Laplace distribution. Furthermore, the difference between any two components,

$$Y^{(k)} - Y^{(j)} \stackrel{d}{=} (m_k - m_j)B + B^{\frac{1}{2}}(Z_k - Z_j), \quad 1 \leq k,\ j \leq n, \tag{4}$$

is again a univariate Laplace distribution. However, it is worth mentioning $Y + a$ does not have the asymmetric Laplace distribution, for $a \neq 0$.

The following lemma indicates that the univariate asymmetric Laplace distribution is a special case of the double exponential distribution, but they are not the same.

Lemma 2.1 *The univariate asymmetric Laplace distribution defined by it's characteristic function in (2) is a special case of the double exponential distribution, because the univariate asymmetric Laplace distribution has the density function*

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \quad p > 0, q > 0,\ p + q = 1,$$

but with

$$p\eta_1 = q\eta_2,$$

and the parameters given by

$$\eta_1 = \frac{2}{\sqrt{m^2 + 2v^2} + m}, \quad \eta_2 = \frac{2}{\sqrt{m^2 + 2v^2} - m}, \quad p = \frac{\sqrt{m^2 + 2v^2} + m}{2\sqrt{m^2 + 2v^2}}. \tag{5}$$

Proof. The characteristic function of a double exponential distribution is given by

$$\Psi_Y(\theta) = \frac{1 + \left(\frac{p}{\eta_2} - \frac{q}{\eta_1}\right) i\theta}{1 + \frac{1}{\eta_1 \eta_2}\theta^2 - \left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) i\theta}.$$

Comparing it with (2) yields that if we set

$$\frac{p}{\eta_2} = \frac{q}{\eta_1}, \quad v = \sqrt{\frac{2}{\eta_1 \eta_2}}, \quad m = \frac{1}{\eta_1} - \frac{1}{\eta_2}, \quad p + q = 1,$$

then the univariate asymmetric Laplace distribution with parameter m and v is actually a double exponential distribution. Solving the above four equations leads to η_1, η_2, p, q as in (5). □

Asymmetric Laplace distribution can also be viewed as a special case of the generalized hyperbolic distribution introduced by Barndorff-Nielsen (1977). In fact, a generalized hyperbolic random variable X is defined as

$$X \stackrel{d}{=} \mu + m\zeta + \zeta^{1/2} Z,$$

where Z is a multivariate normal distribution, ζ is a generalized inverse Gaussian distribution. Since the exponential random variable belongs to generalized inverse Gaussian distribution, the asymmetric Laplace distribution is a special case of the generalized hyperbolic distribution by the representation (3). For more details on applications of the generalized hyperbolic distribution in finance, see Prause and Eberlein (2002).

2.2 A Multivariate Jump-Diffusion Model

We propose a multivariate jump diffusion model in which the asset prices $S(t)$ have two parts, a continuous part driven by a multivariate geometric Brownian motion, and a jump part with jump events modeled by a Poisson process. In the model, there are both common jumps and individual jumps. More precisely, if a Poisson event corresponds to a common jump, then all the asset prices will jump according to the multivariate asymmetric Laplace distribution; otherwise, if a Poisson event corresponds to an individual jump of the j-th asset, then only the j-th asset will jump. In other words, the model attempts to capture various ways of correlated jumps in asset prices.

Mathematically, under the physical measure P the following stochastic differential equation is proposed to model the asset prices $S(t)$:

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right), \tag{6}$$

where $W(t)$ is an n-dimensional standard Brownian motion, $\sigma \in R^{n\times n}$ with the covariance matrix $\Sigma = \sigma\sigma^T$. The rate of the Poisson process $N(t)$ process is $\lambda = \lambda_c + \sum_{k=1}^{n} \lambda_k$; in other words, there are two types of jumps, common jumps for all assets with jump rate λ_c and individual jumps with rate λ_k, $1 \le k \le n$, only for the k-th asset.

The logarithms of the common jumps have a m-dimensional asymmetric Laplace distribution $\mathcal{AL}_n(m_c, J_c)$, where $m_c = (m_{1,c}, ..., m_{n,c})' \in R^n$ and $J_c \in R^{n\times n}$ is positive definite. For the individual jumps of the k-th asset, the logarithms of the jump sizes follow a one-dimensional asymmetric Laplace distribution, $\mathcal{AL}_1(m_k, v_k^2)$. In summary

$$Y = \log(V) \sim \begin{cases} \mathcal{AL}_n(m_c, J_c), & \text{with prob. } \lambda_c/\lambda \\ (\underbrace{0, ..., 0}_{k-1}, \mathcal{AL}_1(m_k, v_k^2), \underbrace{0, ..., 0}_{n-k})', & \text{with prob. } \lambda_k/\lambda, \quad 1 \le k \le n. \end{cases}$$

The sources of randomness, $N(t)$, $W(t)$ are assumed to be independent of the jump sizes V_i's. Jumps at different times are assumed to be independent. Note that in the univariate case, by Lemma 2.1, the above model degenerates to the double exponential jump diffusion model (Kou, 2002) but with $p\eta_1 = q\eta_2$.

Solving the stochastic differential equation in (6) gives the dynamic of the asset prices:

$$S(t) = S(0)\exp\left[(\mu - \frac{1}{2}\Sigma_{diag})t + \sigma W(t)\right]\prod_{i=1}^{N(t)} V_i, \tag{7}$$

where Σ_{diag} denotes the diagonal vector of Σ. Note that $\forall 1 \le k \le n$,

$$\mathsf{E}(V^{(k)}) = \mathsf{E}(e^{Y^{(k)}}) = \frac{\lambda_c/\lambda}{1 - m_{k,c} - J_{c,kk}/2} + \frac{\lambda_k/\lambda}{1 - m_k - v_k^2/2}. \tag{8}$$

The requirements $m_{k,c} + J_{c,kk}/2 < 1$ and $m_k + v_k^2/2 < 1$ are needed to ensure $\mathsf{E}(V^{(k)}) < \infty$ and $\mathsf{E}(S_k(t)) < \infty$, i.e. the stock price has finite expectation.

Since $S(t)$ is a Markov process, an alternative characterization of $S(t)$ is to use the generator of $X_t = \log S(t)/S(0)$. The generator for the two-dimensional case is given in (17), and the multivariate case can be treated similarly.

2.3 A Two-Dimensional Jump Diffusion and Change of Measures

Since the focus of the paper is option pricing for two assets, we shall specialize in the case of two assets from now on. The general finance theory indicates that the price of a European-type option (including path-dependent options, such as barrier options) is given by the discounted payoff at maturity under a risk neutral measure induced by a "pricing kernel". Since the model has jumps, the market is incomplete and there is no unique pricing kernel to choose from. Following Duffie et al. (2000), if we assume the pricing kernel to be an affine asymmetric Laplace jump diffusion process, it can be shown in Appendix A that the asset prices will still behave as an asymmetric Laplace jump diffusion under the induced risk neutral measure P^*. In particular, combining the asset dynamic in (7) with the result of risk-neutral pricing in Theorem A.1 in Appendix A yields that under the risk-neutral measure P^* the asset prices can be written as

$$\begin{aligned} S_1(t) &= S_1(0)\exp\left[\left(r - \frac{1}{2}\sigma_1^2 - \lambda\zeta_1\right)t + \sigma_1 W_1(t) + \sum_{i=1}^{N(t)} Y_i^{(1)}\right], \\ S_2(t) &= S_2(0)\exp\left[\left(r - \frac{1}{2}\sigma_2^2 - \lambda\zeta_2\right)t + \sigma_2[\rho W_1(t) + \sqrt{1-\rho^2}W_2(t)] + \sum_{i=1}^{N(t)} Y_i^{(2)}\right], \end{aligned} \tag{9}$$

with

$$\zeta_k = \mathsf{E}(e^{Y^{(k)}}) - 1 = \frac{\lambda_k/\lambda}{1 - m_k - v_k^2/2} + \frac{\lambda_c/\lambda}{1 - m_{k,c} - v_{k,c}^2/2} - (\lambda_k + \lambda_c)/\lambda, \quad k = 1, 2. \tag{10}$$

Here all the parameters are risk-neutral parameters, $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions, r is the risk-free interest rate, and $N(t)$ is a Poisson process with rate $\lambda =$

$\lambda_c + \lambda_1 + \lambda_2$. The distribution of the logarithm of the jump sizes Y_i is given by

$$Y_i = (Y_i^{(1)}, Y_i^{(2)})' \sim \begin{cases} \mathcal{AL}_2(m_c, J_c), & \text{with prob. } \lambda_c/\lambda \\ (\mathcal{AL}_1(m_1, v_1^2), 0)', & \text{with prob. } \lambda_1/\lambda, \\ (0, \mathcal{AL}_1(m_2, v_2^2))', & \text{with prob. } \lambda_2/\lambda, \end{cases} \tag{11}$$

where the parameters for the common jumps are

$$m_c = \begin{pmatrix} m_{1,c} \\ m_{2,c} \end{pmatrix}, \qquad J_c = \begin{pmatrix} v_{1,c}^2 & cv_{1,c}v_{2,c} \\ cv_{1,c}v_{2,c} & v_{2,c}^2 \end{pmatrix}.$$

Note that

$$S_i(t) = e^{-r(T-t)}\mathsf{E}^*[S_i(T)|\mathcal{F}_t], \quad 0 \le t \le T, \quad i = 1, 2,$$

consistent with the risk-neutrality.

It is useful to discuss change of measures to be used later for option pricing. Introduce an exponential transform of the random vector $Y = (Y^{(1)}, Y^{(2)})'$ in (11), $\varphi(z) = \mathsf{E}(\exp(z'Y))$, $z' = (z_1, z_2)$, whenever the expectation is well defined. By the characteristic function of asymmetric Laplace distribution in (1), we have

$$\varphi(z) = \mathsf{E}(\exp(z'Y)) = (\lambda_c/\lambda)\varphi_c(z) + \sum_{k=1}^{2}(\lambda_k/\lambda)\varphi_k(z_k), \tag{12}$$

where

$$\varphi_c(z) = \frac{1}{1 - m_c'z - z'J_cz/2}, \qquad \varphi_k(z_k) = \frac{1}{1 - m_kz_k - v_k^2z_k^2/2} \tag{13}$$

are the exponential transforms of $\mathcal{AL}_2(\, m_c, J_c)$ and $\mathcal{AL}_1(m_k, v_k^2)$, $k = 1$, 2, respectively.

Lemma 2.2 *Introduce a new probability measure $\bar{\mathsf{P}}$, $\frac{d\bar{\mathsf{P}}}{d\mathsf{P}} = e^{\delta' X_t}/\mathsf{E}[e^{\delta' X_t}]$, where the 2-dimensional process X_t is given by*

$$X_t = X_0 + \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \tag{14}$$

and $\delta = (\delta_1, \delta_2)' \in R^2$ such that $\varphi(\delta) < \infty$. Then under $\bar{\mathsf{P}}$,

$$X_t = X_0 + \bar{\mu}dt + \sigma\bar{W}(t) + \sum_{i=1}^{N(t)} Y_i, \quad \bar{W}(t) = W(t) - (\sigma^T\delta)t, \ \ \bar{\mu} = \mu + \sigma^T\sigma\delta,$$

where $\bar{W}(t)$ becomes a new standard two-dimensional Brownian motion, and the jump size $(Y^{(1)}, Y^{(2)})$ has a new probability density

$$\bar{f}_{(Y^{(1)},Y^{(2)})}(y) = e^{\delta' y} f_{(Y^{(1)},Y^{(2)})}(y)/\varphi(\delta),$$

which still has a mixture of asymmetric bivariate Laplace distributions as in (11) with parameters

$$\bar{m}_c = (m_c + J_c\delta)\varphi_c(\delta), \quad \bar{J}_c = J_c\varphi_c(\delta),$$
$$\bar{m}_k = (m_k + \delta_k v_k^2)\varphi_k(\delta_k), \quad \bar{v}_k = v_k\sqrt{\varphi_k(\delta_k)}, \quad k = 1, 2. \tag{15}$$

The new Poisson process $N(t)$ has a jump rate

$$\bar{\lambda} = \lambda\varphi(\delta) = \bar{\lambda}_c + \bar{\lambda}_1 + \bar{\lambda}_2, \ \bar{\lambda}_c = \lambda_c\varphi_c(\delta), \ \bar{\lambda}_k = \lambda_k\varphi_k(\delta_k), \quad k = 1, 2.$$

Proof. By the Girsanov Theorem for the jump process (Protter, 1990, Schroder, 1999), we know that after change of measure $\bar{W}(t) = W(t) - (\sigma^T\delta)t$ becomes a standard Brownian motion under $\bar{P}$. The diffusion part can be re-written as

$$\mu dt + \sigma dW(t) = \mu dt + \sigma\left(d\bar{W}(t) + \sigma^T\delta dt\right) = (\mu + \sigma\sigma^T\delta)dt + \sigma d\bar{W}(t) = \bar{\mu}dt + \sigma d\bar{W}(t).$$

The new jump rate becomes $\bar{\lambda} = \lambda\varphi(\delta)$ and the jump size has a new probability density

$$\bar{f}_{(Y^{(1)},Y^{(2)})}(y) = e^{\delta' y} f_{(Y^{(1)},Y^{(2)})}(y)/\varphi(\delta).$$

For the new jump density $\bar{f}$, we have that the exponential transform is given by

$$\bar{\varphi}(z) = \int_{R^2} \exp(z'y + \delta' y) f_{(Y^{(1)},Y^{(2)})}(y)/\varphi(\delta)dy = \varphi(z+\delta)/\varphi(\delta).$$

The definition of $\varphi(\cdot)$ in (12) yields

$$\begin{aligned}
\bar{\varphi}(z) &= \frac{\lambda_c/(\lambda\varphi(\delta))}{1 - m_c'(z+\delta) - (z+\delta)'J_c(z+\delta)/2} + \sum_{k=1}^{2}\frac{\lambda_k/(\lambda\varphi(\delta))}{1 - m_k(z_k+\delta_k) - v_k^2(z_k+\delta_k)^2/2} \\
&= \frac{\lambda_c/(\lambda\varphi(\delta))}{\{1/\varphi_c(\delta)\} - (m_c' + \delta' J_c)z - z'J_c z/2} + \sum_{k=1}^{2}\frac{\lambda_k/(\lambda\varphi(\delta))}{\{1/\varphi_k(\delta_k)\} - (m_k + v_k^2\delta_k)z_k - v_k^2 z_k^2/2} \\
&= \frac{\lambda_c\varphi_c(\delta)/(\lambda\varphi(\delta))}{1 - \varphi_c(\delta)(m_c' + \delta' J_c)z - z'J_c z\varphi_c(\delta)/2} + \sum_{k=1}^{2}\frac{\lambda_k\varphi_k(\delta_k)/(\lambda\varphi(\delta))}{1 - \varphi_k(\delta_k)(m_k + v_k^2\delta_k)z_k - v_k^2 z_k^2\varphi_k(\delta_k)/2} \\
&= \frac{\bar{\lambda}_c/\bar{\lambda}}{1 - \bar{m}_c' z - z'\bar{J}_c z/2} + \sum_{k=1}^{2}\frac{\bar{\lambda}_k/\bar{\lambda}}{1 - \bar{m}_k z_k - \bar{v}_k^2 z_k^2/2},
\end{aligned}$$

which is exactly the exponential transform of Y under $\bar{\mathsf{P}}$ with the parameters stated in the lemma. The equality $\lambda\varphi(\delta) = \bar{\lambda}_c + \bar{\lambda}_1 + \bar{\lambda}_2$ follows from (12). □

3 First Passage Times

Given a probability measure P, consider a two-dimensional process $(X_t^{(1)}, X_t^{(2)})$ defined as

$$\begin{aligned} X_t^{(1)} &= \mu_1 t + \sigma_1 W_1(t) + \sum_{i=1}^{N(t)} Y_i^{(1)}, \\ X_t^{(2)} &= \mu_2 t + \sigma_2[\rho W_1(t) + \sqrt{1-\rho^2} W_2(t)] + \sum_{i=1}^{N(t)} Y_i^{(2)}, \end{aligned} \tag{16}$$

where $N(t)$ is a Poisson process with rate $\lambda_c + \lambda_1 + \lambda_2$, $W_1(t)$ and $W_2(t)$ are independent standard Brownian motion, $(Y^{(1)}, Y^{(2)})'$ follows the joint distribution described in (11). In particular, the infinitesimal generator of the two-dimensional process $(X_t^{(1)}, X_t^{(2)})$ jump processes is

$$\begin{aligned} \mathcal{L}u &= \mu_1 \frac{\partial u}{\partial x_1} + \mu_2 \frac{\partial u}{\partial x_2} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 u}{\partial x_2^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ &+\lambda_c \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [u(x_1+y_1, x_2+y_2) - u(x_1,x_2)] f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2) dy_1 dy_2, \\ &+\lambda_1 \int_{-\infty}^{\infty} [u(x_1+y_1, x_2) - u(x_1,x_2)] f_{Y^{(1)}}(y_1) dy_1 \\ &+\lambda_2 \int_{-\infty}^{\infty} [u(x_1, x_2+y_2) - u(x_1,x_2)] f_{Y^{(2)}}(y_2) dy_2, \end{aligned} \tag{17}$$

for all continuously twice differentiable function $u(x_1, x_2)$, where $f^c_{(Y^{(1)},Y^{(2)})}(y_1, y_2)$ is the joint common (correlated) jump density of $\mathcal{AL}_2(m_c, J_c)$ and $f_{Y^{(i)}}(y_i)$ is the individual jump density of $\mathcal{AL}_1(m_i, J_i)$, $i = 1, 2$.

To price two dimensional barrier options, it is essential to compute the joint distribution

$$\mathsf{P}(X_t^{(1)} \le a,\ \max_{0\le s\le t} X_s^{(2)} \ge b) = \mathsf{P}(X_t^{(1)} \le a,\ \tau_b \le t),$$

where the first passage time of process $X_t^{(2)}$ is given by

$$\tau_b \equiv \tau_b^{(2)} := \inf\{t \ge 0 : X_t^{(2)} \ge b\},\ b > 0, .$$

Generally it is difficult to have analytical solution for first passage times under jump models. Here analytical solutions of Laplace transforms of the first passage times are obtained mainly due to the memoryless property of exponential random variables.

We shall report in this section the results for the case of $\lambda_1 > 0$, $\lambda_2 > 0$. The degenerated case of $\lambda_1 = \lambda_2 = 0$ will be treated in Appendix C.

3.1 Some Preliminary Results

We can easily see

$$\mathsf{E}\left[e^{-\theta X_t^{(1)}+\delta X_t^{(2)}}\right] = \exp\{M_e(\theta,\delta)t\}, \;\; \theta>0,\, \delta>0,$$

whenever the expectation exists, where the function $M_e(\cdot,\cdot)$ is defined as

$$\begin{aligned} M_e(x,y) &= -\mu_1 x + \mu_2 y + \frac{1}{2}\sigma_1^2 x^2 + \frac{1}{2}\sigma_2^2 y^2 - \rho\sigma_1\sigma_2 xy \\ &\quad + \frac{\lambda_c}{1 + x m_{1,c} - y m_{2,c} - \frac{1}{2}v_{1,c}^2 x^2 - \frac{1}{2}v_{2,c}^2 y^2 + c v_{1,c} v_{2,c} xy} \\ &\quad + \frac{\lambda_1}{1 + x m_1 - \frac{1}{2}v_1^2 x^2} + \frac{\lambda_2}{1 - y m_2 - \frac{1}{2}v_2^2 y^2} - \lambda_c - \lambda_1 - \lambda_2. \end{aligned} \tag{18}$$

Proposition 3.1 *For any $\alpha > 0$, there exists $\theta_0 > 0$, such that for any fixed $\theta \in (0,\theta_0)$, the equation*

$$M_e(\theta, y) = \alpha,$$

has exactly six distinctive roots: $\beta_i \equiv \beta_{i,\alpha,\theta}$, $i = 1, ..., 6$, three positive and three negative, where

$$0 < \beta_1 < \min(\beta_2^h, \beta_2^g) < \beta_2 < \max(\beta_2^h, \beta_2^g) < \beta_3 < \infty, \tag{19}$$

$$-\infty < \beta_4 < \min(\beta_1^h, \beta_1^g) < \beta_5 < \max(\beta_1^h, \beta_1^g) < \beta_6 < 0. \tag{20}$$

Here β_1^g, β_2^g and β_1^h, β_2^h are defined as

$$\beta_1^g = -\frac{m_{2,c}}{v_{2,c}^2} + \frac{c v_{1,c}\theta}{v_{2,c}} - \frac{\sqrt{L}}{v_{2,c}^2}, \;\; \beta_2^g = -\frac{m_{2,c}}{v_{2,c}^2} + \frac{c v_{1,c}\theta}{v_{2,c}} + \frac{\sqrt{L}}{v_{2,c}^2} \tag{21}$$

$$\beta_1^h = -\frac{m_2}{v_2^2} - \frac{1}{v_2^2}\sqrt{m_2^2 + 2v_2^2}, \qquad \beta_2^h = -\frac{m_2}{v_2^2} + \frac{1}{v_2^2}\sqrt{m_2^2 + 2v_2^2}. \tag{22}$$

where for all small enough $\theta \in (0,\theta_0)$

$$L = m_{2,c}^2 - 2c v_{1,c} v_{2,c} m_{2,c}\theta + 2v_{2,c}^2 + 2v_{2,c}^2 m_{1,c}\theta - \theta^2 v_{1,c}^2 v_{2,c}^2 (1-c^2) > 0.$$

Proof. Let

$$\begin{aligned} g(y) &= 1 + \theta m_{1,c} - y m_{2,c} - \frac{1}{2}v_{1,c}^2\theta^2 - \frac{1}{2}v_{2,c}^2 y^2 + c v_{1,c} v_{2,c}\theta y, \\ h(y) &= 1 - y m_2 - v_2^2 y^2/2. \end{aligned} \tag{23}$$

For small enough $\theta > 0$, $g(y)$ has two distinctive real roots β_1^g and β_2^g, and $h(y)$ has two distinctive real roots β_1^h and β_2^h given in (21) and (22). Thus, we can write $g(y) = -\frac{1}{2}v_{2,c}^2(y-\beta_1^g)(y-\beta_2^g)$

and $h(y) = -\frac{1}{2}v_2^2(y-\beta_1^h)(y-\beta_2^h)$. When θ is small, $1+m_{1,c}\theta - v_{1,c}^2\theta^2/2 > 0$. Hence, we have $\beta_1^g < 0 < \beta_2^g$. Also $m_2 < \sqrt{m_2^2+2v_2^2}$, which implies $\beta_1^h < 0 < \beta_2^h$. Therefore,

$$\begin{aligned} M_e(\theta,y) &= -\mu_1\theta+\mu_2 y+\frac{1}{2}\sigma_1^2\theta^2+\frac{1}{2}\sigma_2^2y^2-\rho\sigma_1\sigma_2\theta y+\frac{\lambda_c}{g(y)}+\frac{\lambda_2}{h(y)}+\frac{\lambda_1}{1+\theta m_1-\frac{1}{2}v_1^2\theta^2}-\lambda \\ &= -\mu_1\theta+\mu_2 y+\frac{1}{2}\sigma_1^2\theta^2+\frac{1}{2}\sigma_2^2y^2-\rho\sigma_1\sigma_2\theta y \\ &\quad +\frac{\lambda_c}{-\frac{1}{2}v_{2,c}^2(y-\beta_1^g)(y-\beta_2^g)}+\frac{\lambda_2}{-\frac{1}{2}v_2^2(y-\beta_1^h)(y-\beta_2^h)}+\frac{\lambda_1}{1+\theta m_1-\frac{1}{2}v_1^2\theta^2}-\lambda. \end{aligned}$$

Note that equation $\beta_1^g = \beta_1^h$ can have at most two real roots in terms of θ. Thus, by continuity, for all small enough positive θ, $\beta_1^g \neq \beta_1^h$. Similarly, $\beta_2^g \neq \beta_2^h$ holds for all small positive θ.

If we divide the real line into six parts

$$\begin{aligned} &(0,\min(\beta_2^h,\beta_2^g)),\ (\min(\beta_2^h,\beta_2^g),\max(\beta_2^h,\beta_2^g)),\ (\max(\beta_2^h,\beta_2^g),\infty), \\ &(-\infty,\min(\beta_1^h,\beta_1^g)),\ (\min(\beta_1^h,\beta_1^g),\max(\beta_1^h,\beta_1^g)),\ (\max(\beta_1^h,\beta_1^g),0), \end{aligned}$$

then we see easily (because $\lambda_c,\lambda_1,\lambda_2>0$)

$$\begin{aligned} &M_e(\theta,+\infty)=+\infty,\quad M_e(\theta,\max(\beta_2^h,\beta_2^g)+)=-\infty,\quad M_e(\theta,\max(\beta_2^h,\beta_2^g)-)=+\infty, \\ &M_e(\theta,\min(\beta_2^h,\beta_2^g)+)=-\infty,\quad M_e(\theta,\min(\beta_2^h,\beta_2^g)-)=+\infty, \\ &M_e(\theta,-\infty)=+\infty,\quad M_e(\theta,\min(\beta_1^h,\beta_1^g)-)=-\infty,\quad M_e(\theta,\min(\beta_1^h,\beta_1^g)+)=+\infty \\ &M_e(\theta,\max(\beta_1^h,\beta_1^g)-)=-\infty,\quad M_e(\theta,\max(\beta_1^h,\beta_1^g)+)=+\infty. \end{aligned}$$

Since

$$M_e(\theta,0) = -\mu_1\theta+\frac{1}{2}\sigma_1^2\theta^2+\frac{\lambda_c}{g(0)}+\frac{\lambda_2}{h(0)}+\frac{\lambda_1}{1+\theta m_1-\frac{1}{2}v_1^2\theta^2}-\lambda \to 0,$$

as $\theta\to 0$, we can choose θ small enough such that $M_e(\theta,0)<\alpha$. Then the equation $M_e(\theta,y)=\alpha$, which is essentially a six-degree polynomial, has exactly six roots as in (19) and (20). □

3.2 Laplace Transforms

Theorem 3.2 *For $\alpha>0$ and all $\theta>0$ small enough, let β_1, β_2 and β_3 be the three positive roots for the equation $M_e(\theta,y)=\alpha$. Then for the jump diffusion processes defined as in (16) we have*

$$\begin{aligned} \mathsf{E}[e^{-\alpha\tau_b}e^{-\theta X^{(1)}_{\tau_b}}1_{\{\tau_b<\infty\}}] &= \frac{\beta_2\beta_3}{(\beta_1-\beta_2)(\beta_1-\beta_3)}\frac{(\beta_1-\beta_2^h)(\beta_1-\beta_2^g)}{\beta_2^h\beta_2^g}e^{-\beta_1 b} \\ &\quad+\frac{\beta_1\beta_3}{(\beta_2-\beta_1)(\beta_2-\beta_3)}\frac{(\beta_2-\beta_2^h)(\beta_2-\beta_2^g)}{\beta_2^h\beta_2^g}e^{-\beta_2 b} \\ &\quad+\frac{\beta_1\beta_2}{(\beta_3-\beta_1)(\beta_3-\beta_2)}\frac{(\beta_3-\beta_2^h)(\beta_3-\beta_2^g)}{\beta_2^h\beta_2^g}e^{-\beta_3 b}, \end{aligned}$$

where β_2^h and β_2^g are given in (21) and (22).

We need the following proposition to prove Theorem 3.2.

Proposition 3.3 *Suppose* $u(x_1, x_2) = e^{-\theta x_1}\phi(x_2)$

$$\phi(x_2) = \begin{cases} 1, & x_2 \geq b \\ A_1 e^{-\beta_1(b-x_2)} + B_1 e^{-\beta_2(b-x_2)} + C_1 e^{-\beta_3(b-x_2)}, & x_2 < b \end{cases}$$

Let $\mathcal{L}(\cdot)$ *be the generator in (17). Then for all* $x_2 < b$,

$$\begin{aligned} -\alpha u + \mathcal{L}u &= A_1 e^{-\theta x_1 - \beta_1(b-x_2)} f(\beta_1) + B_1 e^{-\theta x_1 - \beta_2(b-x_2)} f(\beta_2) + C_1 e^{-\theta x_1 - \beta_3(b-x_2)} f(\beta_3) \\ &+ \frac{\lambda_c e^{-\theta x_1 - \beta_2^g(b-x_2)}}{(\beta_2^g - \beta_1^g)v_2^2/2}\left(\frac{A_1}{\beta_1 - \beta_2^g} + \frac{B_1}{\beta_2 - \beta_2^g} + \frac{C_1}{\beta_3 - \beta_2^g} + \frac{1}{\beta_2^g}\right) \\ &+ \frac{\lambda_2 e^{-\theta x_1 - \beta_2^h(b-x_2)}}{\sqrt{m_2^2 + 2v_2^2}}\left(\frac{A_1}{\beta_1 - \beta_2^h} + \frac{B_1}{\beta_2 - \beta_2^h} + \frac{C_1}{\beta_3 - \beta_2^h} + \frac{1}{\beta_2^h}\right), \end{aligned}$$

where $f(\beta) = M_e(\theta, \beta) - \alpha$ *and* β_2^h, β_2^g *are defined in (21) and (22).*

Proof. See Appendix B. □

Proof of Theorem 3.2. First, we explain heuristically how the parameters in the theorem are obtained. By Proposition 3.1, we know that there exist $\beta_1, \beta_2, \beta_3 > 0$ such that $M_e(\theta, \beta_1) = M_e(\theta, \beta_2) = M_e(\theta, \beta_3) = \alpha$, i.e.$f(\beta_1) = f(\beta_2) = f(\beta_3) = 0$. Solving

$$\frac{A_1}{\beta_1 - \beta_2^g} + \frac{B_1}{\beta_2 - \beta_2^g} + \frac{C_1}{\beta_3 - \beta_2^g} + \frac{1}{\beta_2^g} = 0, \quad \frac{A_1}{\beta_1 - \beta_2^h} + \frac{B_1}{\beta_2 - \beta_2^h} + \frac{C_1}{\beta_3 - \beta_2^h} + \frac{1}{\beta_2^h} = 0,$$

and setting (by appealing to the smooth-fit principle) $u(x_1, b-) = u(x_1, b+)$, i.e. $A_1 + B_1 + C_1 = 1$, we have

$$\begin{aligned} A_1 &= \frac{\beta_2\beta_3}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)} \frac{(\beta_1 - \beta_2^h)(\beta_1 - \beta_2^g)}{\beta_2^h \beta_2^g} > 0, \\ B_1 &= \frac{\beta_1\beta_3}{(\beta_2 - \beta_1)(\beta_2 - \beta_3)} \frac{(\beta_2 - \beta_2^h)(\beta_2 - \beta_2^g)}{\beta_2^h \beta_2^g} > 0 \\ C_1 &= 1 - A_1 - B_1 = \frac{\beta_1\beta_2}{(\beta_3 - \beta_1)(\beta_3 - \beta_2)} \frac{(\beta_3 - \beta_2^h)(\beta_3 - \beta_2^g)}{\beta_2^h \beta_2^g} > 0. \end{aligned}$$

Thus, from Proposition 3.3, with the above chosen parameters we have

$$-\alpha u + \mathcal{L}u = 0, \qquad \forall (x_1, x_2): \quad x_2 < b. \tag{24}$$

Next we will rigorously prove that $u(x_1, x_2)$ is indeed what we want. Because the function $u(x_1, x_2)$ is continuous, but not C^1 at (x_1, b), we cannot apply Itô's formula directly to the process

$e^{-\alpha t}u(X_t^{(1)}, X_t^{(2)})$. However, it is not difficult to see that there exists a sequence of functions $\{\phi_n(x_2);\ n = 1, 2, \cdots\}$,

$$u_n(x_1, x_2) := e^{-\theta x_1}\phi_n(x_2),$$

such that: (1) $\phi_n(x_2)$ is smooth everywhere, and in particular it belongs to C^2; (2) $\phi_n(x_2) = \phi(x_2)$ for all (x_1, x_2) with $x_2 \le b$; (3) $\phi_n(x_2) = 1 = \phi(x_2)$ for all (x_1, x_2) with $x_2 \ge b + \frac{1}{n}$; (4) $0 \le \phi_n(x_2) \le 2$ for all (x_1, x_2) and n. Clearly, $\phi_n(x_2) \to \phi(x_2)$.

For (x_1, x_2) with $x_2 < b$,

$$\begin{aligned}
\mathcal{L}u_n(x_1, x_2) &= \mu_1 \frac{\partial u}{\partial x_1} + \mu_2 \frac{\partial u}{\partial x_2} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 u}{\partial x_2^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \\
&\quad + \lambda_1 \int_{-\infty}^{\infty} u(x_1 + y_1, x_2) f_{Y^{(1)}}(y_1) dy_1 + \lambda_2 \int_{-\infty}^{b-x_2} u(x_1, x_2 + y_2) f_{Y^{(2)}}(y_2) dy_2 \\
&\quad + \lambda_2 \int_{b-x_2}^{b-x_2+\frac{1}{n}} u_n(x_1, x_2 + y_2) f_{Y^{(2)}}(y_2) dy_2 + \lambda_2 \int_{b-x_2+\frac{1}{n}}^{\infty} u(x_1, x_2 + y_2) f_{Y^{(2)}}(y_2) dy_2 \\
&\quad + \lambda_c \int_{-\infty}^{b-x_2} \int_{-\infty}^{\infty} u(x_1 + y_1, x_2 + y_2) f^c_{(Y^{(1)}, Y^{(2)})}(y_1, y_2) dy_1 dy_2 \\
&\quad + \lambda_c \int_{b-x_2}^{b-x_2+\frac{1}{n}} \int_{-\infty}^{\infty} u_n(x_1 + y_1, x_2 + y_2) f^c_{(Y^{(1)}, Y^{(2)})}(y_1, y_2) dy_1 dy_2 \\
&\quad + \lambda_c \int_{b-x_2+\frac{1}{n}}^{\infty} \int_{-\infty}^{\infty} u(x_1 + y_1, x_2 + y_2) f^c_{(Y^{(1)}, Y^{(2)})}(y_1, y_2) dy_1 dy_2.
\end{aligned}$$

By (24),

$$\begin{aligned}
\mathcal{L}u_n(x_1, x_2) &= \alpha u(x_1, x_2) + \lambda_2 \int_{b-x_2}^{b-x_2+\frac{1}{n}} [u_n(x_1, x_2 + y_2) - u(x_1, x_2 + y_2)] f_{Y^{(2)}}(y_2) dy_2, \\
&\quad + \lambda_c \int_{b-x_2}^{b-x_2+\frac{1}{n}} \int_{-\infty}^{\infty} [u_n(x_1 + y_1, x_2 + y_2) - u(x_1 + y_1, x_2 + y_2)] f^c_{(Y^{(1)}, Y^{(2)})}(y_1, y_2) dy_1 dy_2.
\end{aligned}$$

Applying the Itô formula for jump processes (see, e.g., Protter, 1990) to $e^{-\alpha t}u_n(X_t^{(1)}, X_t^{(2)})$, we obtain that the process

$$M_t^{(n)} := e^{-\alpha(t\wedge\tau_b)} u_n(X_{t\wedge\tau_b}^{(1)}, X_{t\wedge\tau_b}^{(2)}) - \int_0^{t\wedge\tau_b} e^{-\alpha s} \left(-\alpha u_n(X_s^{(1)}, X_s^{(2)}) + \mathcal{L}u_n(X_s^{(1)}, X_s^{(2)}) \right) ds, \quad t \ge 0,$$

is a local martingale starting from $M_0^{(n)} = u_n(0, 0) = u(0, 0)$. In particular, there exists a sequence of stopping times $0 \le \xi_m \to \infty$ such that

$$\begin{aligned}
& u(0,0) \qquad (25) \\
= \ & \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b\wedge\xi_m)} u_n(X_{t\wedge\tau_b\wedge\xi_m}) - \int_0^{t\wedge\tau_b\wedge\xi_m} e^{-\alpha s} (-\alpha u_n(X_s^{(1)}, X_s^{(2)}) + \mathcal{L}u_n(X_s^{(1)}, X_s^{(2)}))\, ds \right].
\end{aligned}$$

Since $|\phi_n - \phi| \le 3$ and $\phi_n = \phi$ if $x_2 \le b$ by construction, it follows that, $\forall x_2 < b$,

$$\begin{aligned}
& |-\alpha u_n(x_1,x_2) + \mathcal{L}u_n(x_1,x_2)| \\
\le\ & 3\lambda_c \int_{b-x_2}^{b-x_2+\frac{1}{n}} \int_{-\infty}^{\infty} e^{-\theta(x_1+y_1)} f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2) dy_1 dy_2 + 3\lambda_2 \int_{b-x_2}^{b-x_2+\frac{1}{n}} e^{-\theta x_1} f_{Y^{(2)}}(y_2) dy_2, \\
=\ & 3\lambda_c e^{-\theta x_1} \mathsf{E}[e^{-\theta Y^{(1)}} 1_{\{b-x_2 \le Y^{(2)} \le b-x_2+\frac{1}{n}\}}] + \lambda_2 e^{-\theta x_1} \mathsf{P}[b - x_2 \le Y^{(2)} \le b - x_2 + \frac{1}{n}\}].
\end{aligned}$$

Therefore, by the dominated convergence theorem

$$\lim_{n\to\infty} \mathsf{E}\left[\int_0^{t\wedge\tau_b\wedge\xi_m} e^{-\alpha s}(-\alpha u_n(X_s^{(1)}, X_s^{(2)}) + \mathcal{L}u_n(X_s^{(1)}, X_s^{(2)}))\, ds\right] = 0, \tag{26}$$

because

$$\mathsf{E}\left[\int_0^t e^{-\alpha s}\{\lambda_c e^{-\theta X_s^{(1)}} \mathsf{E}[e^{-\theta Y^{(1)}}] + \lambda_2 e^{-\theta X_s^{(1)}}\} ds\right] < \infty,$$

for θ is small enough so that $\alpha > M_e(\theta, 0)$ and $\mathsf{E}[e^{-\theta Y^{(1)}}] < \infty$. In addition, it follows again from the dominated convergence theorem that

$$\begin{aligned}
& \lim_{n\to\infty} \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b\wedge\xi_m)} u_n\left(X^{(1)}_{t\wedge\tau_b\wedge\xi_m}, X^{(2)}_{t\wedge\tau_b\wedge\xi_m}\right)\right] \qquad (27) \\
=\ & \lim_{n\to\infty} \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b\wedge\xi_m)} e^{-\theta X^{(1)}_{t\wedge\tau_b\wedge\xi_m}} \phi_n\left(X^{(2)}_{t\wedge\tau_b\wedge\xi_m}\right)\right] = \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b\wedge\xi_m)} u(X^{(1)}_{t\wedge\tau_b\wedge\xi_m}, X^{(2)}_{t\wedge\tau_b\wedge\xi_m})\right],
\end{aligned}$$

because $0 \le \phi_n \le 2$.

Combining (26), (27), and (25), we get for any $t \ge 0$,

$$u(0,0) = \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b\wedge\xi_m)} u(X^{(1)}_{t\wedge\tau_b\wedge\xi_m}, X^{(2)}_{t\wedge\tau_b\wedge\xi_m})\right].$$

Since we can find a $\varepsilon > 1$ such that

$$M_e(\theta\varepsilon, 0) < \alpha < \alpha\varepsilon,$$

we have

$$\begin{aligned}
0 \ \le\ & \mathsf{E}\left[\{e^{-\alpha(t\wedge\tau_b\wedge\xi_m)} u(X^{(1)}_{t\wedge\tau_b\wedge\xi_m}, X^{(2)}_{t\wedge\tau_b\wedge\xi_m})\}^\varepsilon\right] \\
\le\ & \mathsf{E}\left[e^{-\alpha\varepsilon(t\wedge\tau_b\wedge\xi_m)} e^{-\theta\varepsilon X^{(1)}_{t\wedge\tau_b\wedge\xi_m}}\right] \\
=\ & \exp\{\{M_e(\varepsilon\theta,0) - \alpha\varepsilon\}(t\wedge\tau_b\wedge\xi_m)\} \mathsf{E}\left[\exp\{-M_e(\varepsilon\theta,0)(t\wedge\tau_b\wedge\xi_m) - \theta\varepsilon X^{(1)}_{t\wedge\tau_b\wedge\xi_m}\}\right] \\
\le\ & \exp\{\{M_e(\varepsilon\theta,0) - \alpha\varepsilon\}t\}.
\end{aligned}$$

Thus,

$$0 \le \mathsf{E}\left[\{e^{-\alpha(t\wedge\tau_b\wedge\xi_m)} u(X^{(1)}_{t\wedge\tau_b\wedge\xi_m}, X^{(2)}_{t\wedge\tau_b\wedge\xi_m})\}^\varepsilon\right] \le 1, \tag{28}$$

irrespective of m and b. The uniform integrability with respect to the index m is obtained by (28); and letting $m \to \infty$ we have

$$\begin{aligned} u(0,0) &= \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b)}u(X^{(1)}_{t\wedge\tau_b}, X^{(2)}_{t\wedge\tau_b})\right] \\ &= \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b)}u(X^{(1)}_{t\wedge\tau_b}, X^{(2)}_{t\wedge\tau_b})1_{\{\tau_b<\infty\}}\right] + \mathsf{E}\left[e^{-\alpha t}u(X^{(1)}_t, X^{(2)}_t)1_{\{\tau_b=\infty\}}\right]. \end{aligned}$$

If we let $t \to \infty$, the second term in the above equation becomes

$$\begin{aligned} 0 &\leq \lim_{t\to\infty} \mathsf{E}\left[e^{-\alpha t}u(X^{(1)}_t, X^{(2)}_t)1_{\{\tau_b=\infty\}}\right] \\ &\leq \lim_{t\to\infty} \mathsf{E}\left[e^{-\alpha t}\exp(-\theta X^{(1)}_t)\right] = \lim_{t\to\infty} \exp\{[-\alpha + M_e(\theta,0)]t\} = 0, \end{aligned}$$

because $0 \leq u(x_1,x_2) \leq e^{-\theta x_1}$ and we can choose θ small enough so that $M_e(\theta,0) < \alpha$; for the first term the uniform integrability (with respect to t) implied from (28) yields

$$\lim_{t\to\infty} \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b)}u(X^{(1)}_{t\wedge\tau_b}, X^{(2)}_{t\wedge\tau_b})1_{\{\tau_b<\infty\}}\right] = \mathsf{E}\left[e^{-\alpha\tau_b-\theta X^{(1)}_{\tau_b}}\phi(X^{(2)}_{\tau_b})1_{\{\tau_b<\infty\}}\right].$$

Therefore, we have

$$u(0,0) = \mathsf{E}\left[e^{-\alpha\tau_b-\theta X^{(1)}_{\tau_b}}1_{\{\tau_b<\infty\}}\right],$$

as $\phi(X^{(2)}_{\tau_b}) = 1$ on set $\{\tau_b < \infty\}$, from which the result follows. □

Next theorem gives the Laplace transform of the joint distribution of the jump diffusion process $X^{(1)}_t$ and first passage time τ_b.

Theorem 3.4 *For any two jump diffusion processes defined as in (16), denote*

$$\Omega_e(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho,\lambda_c,m_{1,c},m_{2,c},v_{1,c},v_{2,c},c,\lambda_1,m_1,v_1,\lambda_2,m_2,v_2;h,b,t) = \mathsf{P}(X^{(1)}_t \leq h, \tau_b \leq t).$$

Then for all $\theta > 0$ small enough such that $\alpha > M_e(\theta,0)$, we have

$$\int_{-\infty}^{\infty} e^{-\theta a}\int_0^{\infty} e^{-\alpha t}\mathsf{P}(X^{(1)}_t \leq h, \tau_b \leq t)dtdh = \frac{1}{\theta(\alpha - M_e(\theta,0))}\mathsf{E}[e^{-\alpha\tau_b}e^{-\theta X^{(1)}_{\tau_b}}1_{\{\tau_b<\infty\}}],$$

where $M_e(\cdot,\cdot)$ is given in (18).

Proof. By Fubini theorem

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-\theta h}\int_0^{\infty} e^{-\alpha t}\mathsf{P}(X^{(1)}_t \leq h,\, \tau_b \leq t)dtdh \\ &= \mathsf{E}\left\{\int_{-\infty}^{\infty} e^{-\theta a}\int_0^{\infty} e^{-\alpha t}1_{\{X^{(1)}_t\leq h\}}1_{\{\tau_b\leq t\}}dtdh\right\} = \mathsf{E}\left\{\int_{\tau_b}^{\infty}\int_{X^{(1)}_t}^{\infty} e^{-\theta h}dh\cdot e^{-\alpha t}dt\right\} \\ &= \mathsf{E}\left\{\int_{\tau_b}^{\infty}\frac{e^{-\theta X^{(1)}_t}}{\theta}e^{-\alpha t}dt\right\} = \frac{1}{\theta}\mathsf{E}\left\{e^{-\alpha\tau_b}\int_0^{\infty} e^{-\theta X^{(1)}_{u+\tau_b}}e^{-au}du\right\}. \end{aligned}$$

Now the strong Markov property yields the conditional expectation

$$\begin{aligned}
&\mathsf{E}\left\{e^{-\alpha\tau_b}\int_0^\infty e^{-\theta X^{(1)}_{u+\tau_b}}e^{-au}du|\mathcal{F}_{\tau_b}\right\}\\
=\ & e^{-\alpha\tau_b}e^{-\theta X^{(1)}_{\tau_b}}\mathsf{E}\left\{\int_0^\infty e^{-\theta X^{(1)}_u}e^{-au}du\right\} = e^{-\alpha\tau_b}e^{-\theta X^{(1)}_{\tau_b}}\int_0^\infty e^{M_e(\theta,0)u}e^{-au}du = \frac{e^{-\alpha\tau_b}e^{-\theta X^{(1)}_{\tau_b}}}{\alpha - M_e(\theta,0)},
\end{aligned}$$

as $\alpha > M_e(\theta, 0)$, from which the conclusion follows. □

4 Pricing Barrier Options

Since all the eight types of barrier option can be solved in similar ways, we shall only illustrate the up-and-in put option.

Theorem 4.1 *The price of a two-dimensional up-and-in put barrier option is given by*

$$\begin{aligned}
&\mathsf{E}^*(e^{-rT}(K-S_1(T))^+1_{\{\max_{0\le t\le T}S_2(t)\ge H\}})\\
=\ & Ke^{-rT}\Omega_e(r-\frac{1}{2}\sigma_1^2-\lambda\zeta_1, r-\frac{1}{2}\sigma_2^2-\lambda\zeta_2,\sigma_1,\sigma_2,\lambda_c,m_{1,c},m_{2,c},v_{1,c},v_{2,c},c,\\
&\lambda_1,m_1,v_1,\lambda_2,m_2,v_2;\log(K/S_1(0)),\log(H/S_2(0)),T)\\
&-S_1(0)\Omega_e(r+\frac{1}{2}\sigma_1^2-\lambda\zeta_1, r-\frac{1}{2}\sigma_2^2+\rho\sigma_1\sigma_2-\lambda\zeta_2,\sigma_1,\sigma_2,\widehat{\lambda}_c,\widehat{m}_{1,c},\widehat{m}_{2,c},\widehat{v}_{1,c},\widehat{v}_{2,c},\widehat{c}, \\
&\widehat{\lambda_1},\widehat{m}_1,\widehat{v}_1,\lambda_2,m_2,v_2;\log(K/S_1(0)),\log(H/S_2(0)),T),
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
&\widehat{m}_1 = \frac{m_1+v_1^2}{1-m_1-v_1^2/2},\quad \widehat{v}_1 = \frac{v_1}{\sqrt{1-m_1-v_1^2/2}},\quad \widehat{\lambda}_1 = \frac{\lambda_1}{1-m_1-v_1^2/2},\\
&\widehat{m}_{1,c} = \frac{m_{1,c}+v_{1,c}^2}{1-m_{1,c}-v_{1,c}^2/2},\quad \widehat{m}_{2,c} = \frac{m_{2,c}+cv_{1,c}v_{2,c}}{1-m_{1,c}-v_{1,c}^2/2},\quad \widehat{c}=c,\\
&\widehat{v}_{1,c} = \frac{v_{1,c}}{\sqrt{1-m_{1,c}-v_{1,c}^2/2}},\quad \widehat{v}_{2,c} = \frac{v_{2,c}}{\sqrt{1-m_{1,c}-v_{1,c}^2/2}},\quad \widehat{\lambda}_c = \frac{\lambda_c}{1-m_{1,c}-v_{1,c}^2/2},
\end{aligned}$$

ζ_k *is given in (10), for* $k=1,2$, *and the function* Ω_e *is defined in Theorem 3.4.*

Proof. The price of a two-dimensional up-and-in put barrier option is given by

$$\begin{aligned}
&\mathsf{E}^*(e^{-rT}(K-S_1(T))1_{\{S_1(T)\le K,\max_{0\le t\le T}S_2(t)\ge H\}})\\
=\ & Ke^{-rT}\mathsf{P}(S_1(T)\le K,\max_{0\le t\le T}S_2(t)\ge H) - e^{-rT}\mathsf{E}^*(S_1(T)1_{\{S_1(T)\le K,\max_{0\le t\le T}S_2(t)\ge H\}}).
\end{aligned}$$

The probability in the first term can be calculated directly from the function Ω_e. Hence, we only need to calculate the second term. Using the probability measure $\widehat{\mathsf{P}}$,

$$\frac{d\widehat{\mathsf{P}}}{d\mathsf{P}^*}|_{t=T} := \frac{e^{-rT}S_1(T)}{S_1(0)} = \exp\left[\left(-\frac{1}{2}\sigma_1^2 - \lambda\zeta_1\right)T + \sigma_1 W_1(T) + \sum_{i=1}^{N(T)} Y_i^{(1)}\right],$$

we have

$$e^{-rT}\mathsf{E}^*(S_1(T)1_{\{S_1(T)\leq K, \max_{0\leq t\leq T} S_2(t)\geq H\}}) = S_1(0)\widehat{\mathsf{P}}(S_1(T)\leq K, \max_{0\leq t\leq T} S_2(t)\geq H).$$

Using Lemma 2.2 with $\delta = (1,0)'$, we have under new measure $\widehat{\mathsf{P}}$, $S_1(t)$ and $S_2(t)$ have the following dynamic:

$$\begin{aligned} S_1(t) &= S_1(0)\exp\left[\left(r + \frac{1}{2}\sigma_1^2 - \lambda\zeta_1\right)t + \sigma_1\widehat{W}_1(t) + \sum_{i=1}^{N(t)} Y_i^{(1)}\right], \\ S_2(t) &= S_2(0)\exp\left[\left(r - \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 - \lambda\zeta_2\right)t + \sigma_2[\rho\widehat{W}_1(t) + \sqrt{1-\rho^2}\widehat{W}_2(t)] + \sum_{i=1}^{N(t)} Y_i^{(2)}\right], \end{aligned}$$

where $\widehat{W}_1(t)$ and $\widehat{W}_2(t)$ are two new standard Brownian motions. The Poisson process $N(t)$ has a new rate

$$\widehat{\lambda} = \lambda\mathsf{E}(e^{Y^{(1)}}) = \lambda(1+\zeta_1) = \frac{\lambda_c}{1 - m_{1,c} - v_{1,c}^2/2} + \frac{\lambda_1}{1 - m_1 - v_1^2/2} + \lambda_2,$$

$$\widehat{\lambda}_c = \frac{\lambda_c}{1 - m_{1,c} - v_{1,c}^2/2}, \quad \widehat{\lambda}_1 = \frac{\lambda_1}{1 - m_1 - v_1^2/2}, \quad \widehat{\lambda}_2 = \lambda_2.$$

The jump sizes $(Y_i^{(1)}, Y_i^{(2)})$ has a distribution

$$(Y_i^{(1)}, Y_i^{(2)})' \sim \begin{cases} \mathcal{AL}_2\left(\begin{pmatrix} \widehat{m}_{1,c} \\ \widehat{m}_{2,c} \end{pmatrix}, \begin{pmatrix} \widehat{v}_{1,c}^2 & c\widehat{v}_{1,c}\widehat{v}_{2,c} \\ c\widehat{v}_{1,c}\widehat{v}_{2,c} & \widehat{v}_{2,c}^2 \end{pmatrix}\right) & w.p.\ \widehat{\lambda}_c/\widehat{\lambda} \\ (\mathcal{AL}_1(\widehat{m}_1, \widehat{v}_1^2), 0)' & w.p.\ \widehat{\lambda}_1/\widehat{\lambda} \\ (0, \mathcal{AL}_1(m_2, v_2^2))' & w.p.\ \lambda_2/\widehat{\lambda} \end{cases}$$

Thus, $\widehat{\mathsf{P}}(S_1(T)\leq K, \max_{0\leq t\leq T} S_2(t)\geq H)$ is still the Ω_e function with parameters stated in (29), from which the theorem follows. □

To compute prices for two-dimensional barrier options, it is enough to compute $\Omega_e(\cdot)$ by using a two-sided, two dimensional Euler algorithm for Laplace transform inversion (Abate and Whitt, 1992, Petrella, 2004). The results are given in Table 1. All the computations are done by a Pentium 750 PC. In the Monte Carlo simulation we use 100,000 replications. To reduce the discretization error resulting from simulating continuous-time processes, we use 2000 and 4000 discretization grids

		$\lambda_1 = \lambda_2 = 0$		$\lambda_1 = 0.5,\ \lambda_2 = 1.5$	
		Analytical	Monte Carlo	Analytical	Monte Carlo
$\lambda_c = 0.01$	$\rho = 0.2$	3.0281	2.9707	2.2155	2.2104
	$c = 0.8$		95% CI:(2.86,3.08)		95% CI:(2.17,2.25)
	$\rho = 0.7$	1.9853	1.9357	1.8625	1.8367
	$c = -0.4$		95% CI:(1.91,1.96)		95% CI:(1.81,1.87)
	$\rho = -0.6$	4.7144	4.6757	2.8386	2.8084
	$c = 0.3$		95% CI:(4.63,4.72)		95% CI:(2.77,2.85)
	$\rho = -0.9$	5.3818	5.3390	3.0756	3.0496
	$c = -0.8$		95% CI:(5.29,5.39)		95% CI:(3.00,3.09)
$\lambda_c = 3.00$	$\rho = 0.2$	9.4453	9.3896	7.3533	7.3257
	$c = 0.8$		95% CI:(9.17,9.60)		95% CI:(7.23,7.42)
	$\rho = 0.7$	10.6796	10.6455	8.5787	8.5041
	$c = -0.4$		95% CI:(10.54,10.75)		95% CI:(8.40,8.60)
	$\rho = -0.6$	10.6105	10.5828	8.1489	8.0972
	$c = 0.3$		95% CI:(10.47,10.69)		95% CI:(8.00,8.19)
	$\rho = -0.9$	12.2657	12.2163	9.5180	9.4734
	$c = -0.8$		95% CI:(12.10,12.33)		95% CI:(9.37,9.58)
Average CPU Time Per Price		2 Min	6 Hr	4 Min	13 Hr

Table 1: Numerical results for the up-and-in-put option. The parameters used are $S_1(0) = 70$, $S_2(0) = 52$, $T = 0.25$, $r = 0.05$, $\sigma_1 = 0.3$, $\sigma_2 = 0.5$, $m_1 = -0.2$, $m_2 = 0.4$, $v_1 = 0.5$, $v_2 = 0.3$, $m_{1,c} = -0.5$, $m_{2,c} = 0.1$, $v_{1,c} = 0.4$, $v_{2,c} = 0.6$, the strike price $K = 60$ and barrier $H = 65$. For the Monte Carlo results, 100,000 simulation runs are used. To reduce the discretization error in simulation, we use 2000 and 4000 discretization grids and then use Richardson's extrapolation for the square root convergence rate. As a comparison, for the Brownian model (i.e. $\lambda_1 = \lambda_2 = \lambda_c = 0$) the option prices are 2.9884, 1.9431, 4.6756, and 5.3428 for $\rho = 0.2$, 0.7, -0.6, -0.9, respectively.

and then use Richardson's extrapolation for the square root convergence rate. As we can see in Table 1 the Monte Carlo results are always smaller than the results from Laplace inversion for the up-and-in put, because the discrete maxima is always smaller than the continuous maxima. To verify the accuracy of the two-dimensional Euler inversion, we also report the results for the Brownian case as a comparison. The table seems to suggest that the analytical formulae lead to accurate numerical results.

5 Pricing of Exchange Options and Related Extensions

5.1 The General Case

We shall study the general case $\lambda_i \geq 0$, $i = 1, 2$ first, which requires Laplace transforms, and discuss the special case of $\lambda_i = 0$, $i = 1, 2$ in the next subsection which does not require Laplace transforms. An exchange option gives the holder the right but not the obligation to exchange shares of one asset (S_2) to shares of another one (S_1) at the expiration date T. The payoff is $(S_1(T) - e^{-k}S_2(T))^+$, for $k \in (-\infty, \infty)$, where e^{-k} is the ratio of shares. The price of such an exchange option is therefore given by

$$\begin{aligned} u(0,k) &= \mathsf{E}^*[e^{-rT}(S_1(T) - e^{-k}S_2(T))^+] \\ &= S_2(0)\mathsf{E}^*\left[\frac{e^{-rT}S_2(T)}{S_2(0)}\left(\frac{S_1(T)}{S_2(T)} - e^{-k}\right)^+\right] \\ &= S_2(0)\widetilde{\mathsf{E}}\left[\left(\frac{S_1(T)}{S_2(T)} - e^{-k}\right)^+\right] = S_2(0)\widetilde{\mathsf{E}}\left[\left(F(T) - e^{-k}\right)^+\right], \end{aligned} \tag{30}$$

where

$$F(T) := \frac{S_1(T)}{S_2(T)} \equiv \frac{S_1(0)}{S_2(0)}\exp[A(T)],$$

and $\widetilde{\mathsf{E}}$ is the expectation taken with respect to the probability $\widetilde{\mathsf{P}}$,

$$\frac{d\widetilde{\mathsf{P}}}{d\mathsf{P}^*}|_{t=T} := \frac{e^{-rT}S_2(T)}{S_2(0)} = \exp\left[\left(-\frac{1}{2}\sigma_2^2 - \lambda\zeta_2\right)T + \sigma_2[\rho W_1(T) + \sqrt{1-\rho^2}W_2(T)] + \sum_{i=1}^{N(T)} Y_i^{(2)}\right].$$

Note that $\widetilde{\mathsf{P}}$ is a well-defined probability as $\mathsf{E}^*\left[\frac{e^{-rT}S_2(T)}{S_2(0)}\right] = 1$.

Theorem 5.1 *The two-sided Laplace transform of $u(0,k)$ with respect to k is given by*

$$\int_{-\infty}^{\infty} e^{-\xi k}u(0,k)dk = S_2(0) \cdot \frac{(S_1(0)/S_2(0))^{\xi+1}}{\xi(\xi+1)}\exp[Q(\xi+1)T], \tag{31}$$

for all $\xi > 0$ small enough such that $\widetilde{\mathsf{E}}[e^{\xi A(T)}] < \infty$, where $Q(\theta)$ is the function such that $\widetilde{\mathsf{E}}[e^{\theta A(T)}] =$

$\exp\{Q(\theta)T\}$; *more precisely,*

$$Q(\theta) = \left(-\frac{1}{2}\sigma^2 - \lambda(\zeta_1 - \zeta_2)\right)\theta + \frac{1}{2}\sigma^2\theta^2 - \lambda_1 - \widetilde{\lambda}_2 - \widetilde{\lambda}_c + \frac{\lambda_1}{1 - m_1\theta - \frac{1}{2}v_1^2\theta^2} + \frac{\widetilde{\lambda}_2}{1 + \widetilde{m}_2\theta - \frac{1}{2}\widetilde{v}_2^2\theta^2} + \frac{\widetilde{\lambda}_c}{1 - \widetilde{m}\theta - \frac{1}{2}\widetilde{v}^2\theta^2}, \tag{32}$$

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \quad \widetilde{\lambda}_c = \frac{\lambda_c}{1 - m_{2,c} - v_{2,c}^2/2}, \tag{33}$$

$$\widetilde{m}_2 = \frac{m_2 + v_2^2}{1 - m_2 - v_2^2/2}, \quad \widetilde{v}_2 = \frac{v_2}{\sqrt{1 - m_2 - v_2^2/2}}, \quad \widetilde{\lambda}_2 = \frac{\lambda_2}{1 - m_2 - v_2^2/2}, \tag{34}$$

$$\widetilde{m} = \frac{m_{1,c} - m_{2,c} + cv_{1,c}v_{2,c} - v_{2,c}^2}{1 - m_{2,c} - v_{2,c}^2/2}, \quad \widetilde{v} = \sqrt{\frac{v_{1,c}^2 + v_{2,c}^2 - 2cv_{1,c}v_{2,c}}{1 - m_{2,c} - v_{2,c}^2/2}}. \tag{35}$$

Proof. Setting $\beta = (0,1)'$ in Lemma 2.2, we get from the representation (9) that under the new measure $\widetilde{\mathsf{P}}$,

$$S_1(T) = S_1(0)\exp\left[\left(r - \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 - \lambda\zeta_1\right)T + \sigma_1\widetilde{W}_1(T) + \sum_{i=1}^{N(T)} Y_i^{(1)}\right]$$

$$S_2(T) = S_2(0)\exp\left[\left(r + \frac{1}{2}\sigma_2^2 - \lambda\zeta_2\right)T + \sigma_2(\rho\widetilde{W}_1(T) + \sqrt{1-\rho^2}\widetilde{W}_2(T)) + \sum_{i=1}^{N(T)} Y_i^{(2)}\right],$$

where $(\widetilde{W}_1(t), \widetilde{W}_2(t))$ forms a two-dimensional standard Brownian motion under $\widetilde{\mathsf{P}}$. The Poisson process $N(t)$ has a new rate

$$\widetilde{\lambda} = \lambda \mathsf{E}^*(e^{Y^{(2)}}) = \lambda(1 + \zeta_2) = \frac{\lambda_c}{1 - m_{2,c} - v_{2,c}^2/2} + \lambda_1 + \frac{\lambda_2}{1 - m_2 - v_2^2/2},$$

via the definition of ζ_2 in (10). Equation (15) implies that $(Y_i^{(1)}, Y_i^{(2)})$ is still an asymmetric Laplace distribution with the new parameters $\widetilde{m}_2$, $\widetilde{v}_2$, $\widetilde{\lambda}_2$, and $\widetilde{\lambda}_c$ as in (33), (34), (35) and

$$\widetilde{m}_1 = m_1, \quad \widetilde{v}_2 = v_1, \quad \widetilde{\lambda}_1 = \lambda_1,$$

$$\widetilde{m}_{1,c} = \frac{m_{1,c} + cv_{1,c}v_{2,c}}{1 - m_{2,c} - v_{2,c}^2/2}, \quad \widetilde{m}_{2,c} = \frac{m_{2,c} + v_{2,c}^2}{1 - m_{2,c} - v_{2,c}^2/2}, \quad \widetilde{c} = c,$$

$$\widetilde{v}_{1,c} = \frac{v_{1,c}}{\sqrt{1 - m_{2,c} - v_{2,c}^2/2}}, \quad \widetilde{v}_{2,c} = \frac{v_{2,c}}{\sqrt{1 - m_{2,c} - v_{2,c}^2/2}}.$$

Hence, we can re-write $A(T)$ as

$$A(T) = \left(-\frac{1}{2}\sigma^2 - \lambda(\zeta_1 - \zeta_2)\right)T + \sigma\widetilde{W}(T) + \sum_{i=1}^{N(T)} Y_i, \quad Y_i = Y_i^{(1)} - Y_i^{(2)},$$

$$\sigma = \sqrt{(\sigma_1 - \sigma_2\rho)^2 + (\sigma_2\sqrt{1-\rho^2})^2} = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$

where

$$\widetilde{W}(T) := \frac{1}{\sigma}\left[(\sigma_1 - \sigma_2\rho)\widetilde{W}_1(T) - \sigma_2\sqrt{1-\rho^2}\widetilde{W}_2(T)\right]$$

is a one-dimensional standard Brownian motion under $\widetilde{\mathsf{P}}$. Using Kotz et al. (2001, pp. 240-241) we know if $(Y_i^{(1)}, Y_i^{(2)})' \sim \mathcal{AL}_2(\begin{pmatrix} \widetilde{m}_{1,c} \\ \widetilde{m}_{2,c} \end{pmatrix}, \begin{pmatrix} \widetilde{v}_{1,c}^2 & \widetilde{c}\widetilde{v}_{1,c}\widetilde{v}_{2,c} \\ \widetilde{c}\widetilde{v}_{1,c}\widetilde{v}_{2,c} & \widetilde{v}_{2,c}^2 \end{pmatrix})$ then $Y_i = Y_i^{(1)} - Y_i^{(2)}$ is a one-dimensional asymmetric Laplace distribution with the parameters

$$\begin{aligned}
\widetilde{v} &= \sqrt{\widetilde{v}_{1,c}^2 - 2\widetilde{c}\widetilde{v}_{1,c}\widetilde{v}_{2,c} + \widetilde{v}_{2,c}^2} = \sqrt{\frac{v_{1,c}^2 - 2cv_{1,c}v_{2,c} + v_{2,c}^2}{1 - m_{2,c} - v_{2,c}^2/2}}, \\
\widetilde{m} &= \widetilde{m}_{1,c} - \widetilde{m}_{2,c} = \frac{m_{1,c} - m_{2,c} + cv_{1,c}v_{2,c} - v_{2,c}^2}{1 - m_{2,c} - v_{2,c}^2/2}.
\end{aligned}$$

In addition, $-\mathcal{AL}_1(m, v^2) \stackrel{d}{=} \mathcal{AL}_1(-m, v^2)$. Therefore, under $\widetilde{\mathsf{P}}$,

$$Y_i = Y_i^{(1)} - Y_i^{(2)} \sim \begin{cases} \mathcal{AL}_1(\widetilde{m}, \widetilde{v}^2), & \text{with prob. } \widetilde{\lambda}_c/\widetilde{\lambda}, \\ \mathcal{AL}_1(m_1, v_1^2), & \text{with prob. } \lambda_1/\widetilde{\lambda}, \\ \mathcal{AL}_1(-\widetilde{m}_2, \widetilde{v}_2^2), & \text{with prob. } \widetilde{\lambda}_2/\widetilde{\lambda}. \end{cases} \tag{36}$$

After determining the dynamics of $A(T)$ under $\widetilde{\mathsf{P}}$ from (30) we know that we only need to calculate $\widetilde{\mathsf{E}}[(F(T) - e^{-k})^+]$. From Kou, Petrella, and Wang (2005), the Laplace transform of $\widetilde{\mathsf{E}}[(F(T) - e^{-k})^+]$ with respect to k is given by

$$\frac{(S_1(0)/S_2(0))^{\xi+1}}{\xi(\xi+1)} \exp[Q(\xi+1)T],$$

where $Q(\theta)$ is the function such that $\widetilde{\mathsf{E}}[e^{\theta A(T)}] = \exp\{Q(\theta)T\}$, from which $Q(\theta)$ can be computed explicitly as in (32) by using the dynamics of $A(T)$ under $\widetilde{\mathsf{P}}$. □

Remark: Immediately from the exchange option prices we can also calculate the prices of the options on the minimum or maximum of two assets (called a "chooser" option), which are given by

$$e^{-rT}\mathsf{E}^*[\min(S_1(T), S_2(T))], \quad e^{-rT}\mathsf{E}^*[\max(S_1(T), S_2(T))],$$

respectively. More precisely, since $\max[x_1, x_2] = (x_1 - x_2)^+ + x_2$ we have

$$\begin{aligned}
e^{-rT}\mathsf{E}^*[\max(S_1(T), S_2(T))] &= e^{-rT}\mathsf{E}^*[(S_1(T) - S_2(T))^+] + e^{-rT}\mathsf{E}[S_2(T)] \\
&= e^{-rT}\mathsf{E}^*[(S_1(T) - S_2(T))^+] + S_2(0).
\end{aligned}$$

Similarly, since $\min[x_1, x_2] = x_1 - (x_1 - x_2)^+$ we have

$$\begin{aligned}
e^{-rT}\mathsf{E}^*[\min(S_1(T), S_2(T))] &= e^{-rT}\mathsf{E}^*[S_1(T)] - e^{-rT}\mathsf{E}[(S_1(T) - S_2(T))^+] \\
&= S_1(0) - e^{-rT}\mathsf{E}^*[(S_1(T) - S_2(T))^+].
\end{aligned}$$

5.2 Special Case $\lambda_1 = \lambda_2 = 0$ without Laplace Transforms

In the case of $\lambda_1 = \lambda_2 = 0$, i.e. when all jumps are common (correlated) jumps and there are no individual jumps, Laplace transforms are not needed. More precisely, for any given probability P, let

$$\Upsilon(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, t) := \mathsf{P}(X_t \geq a),$$

where under P, $X_t = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$, and jump size Y_i has a double exponential distribution with density $f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}}$. The Υ function can be computed explicitly by using the Hh function (Theorem B.1 in Kou 2002). The following Proposition gives the pricing formula of the exchange option in terms of the function Υ.

Proposition 5.2 *For the dynamic as in (9) with $\lambda_1 = \lambda_2 = 0$, the price of exchange option on the two assets is obtained as*

$$\begin{aligned} u(0,k) \quad = \quad & S_1(0)\Upsilon(\frac{1}{2}\sigma^2 - \lambda(\zeta_1 - \zeta_2), \sigma, \tilde{\tilde{\lambda}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; -k + \log(S_2(0)/S_1(0)), T) \\ & -e^{-k} S_2(0)\Upsilon(-\frac{1}{2}\sigma^2 - \lambda(\zeta_1 - \zeta_2), \sigma, \tilde{\lambda}, p, \eta_1, \eta_2; -k + \log(S_2(0)/S_1(0)), T), \end{aligned}$$

where

$$\begin{aligned} &\eta_1 = \frac{2}{\sqrt{\tilde{m}^2 + 2\tilde{v}^2} + \tilde{m}}, \quad \eta_2 = \frac{2}{\sqrt{\tilde{m}^2 + 2\tilde{v}^2} - \tilde{m}}, \quad p = \frac{\sqrt{\tilde{m}^2 + 2\tilde{v}^2} + \tilde{m}}{2\sqrt{\tilde{m}^2 + 2\tilde{v}^2}}, \\ &\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \quad \tilde{\lambda} = \frac{\lambda}{1 - m_2 - v_2^2/2}, \quad \tilde{\tilde{\lambda}} = \tilde{\lambda}(1+\zeta), \\ &\tilde{p} = \frac{p}{1+\zeta} \cdot \frac{\eta_1}{\eta_1 - 1}, \quad \tilde{\eta}_1 = \eta_1 - 1, \quad \tilde{\eta}_2 = \eta_2 + 1, \quad \zeta = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1, \end{aligned}$$

with ζ_1, ζ_2 given in (10) and $\tilde{m}, \tilde{v}$ given in (35).

Proof. If $\lambda_1 = \lambda_2 = 0$, (36) implies that under the measure $\tilde{P}$, $Y_i \sim \mathcal{AL}_1(\tilde{m}, \tilde{v}^2)$. Replacing the m, v in (5) by $\tilde{m}, \tilde{v}$, and plugging in the definition of $\tilde{m}, \tilde{v}$, we get the parameters p, η_1, η_2 in the double exponential distribution. Since (30) implies that the exchange option can be written as an European call option with strike e^{-k} and zero risk-free rate, the result follows immediately from Theorem 2 in Kou (2002). □

5.3 Numerical Results

To implement the pricing formulae, we use the one-dimensional (two-sided) Euler algorithm for the inversion of Laplace transforms; see Abate and Whitt (1992), Choudhury et al. (1994), and Petrella (2004) for the inversion details. In particular, since the exchange options can be reduced to call options, the inversion is identical to that in Petrella (2004).

		$\lambda_1 = \lambda_2 = 0$		$\lambda_1 = 0.5,\ \lambda_2 = 1.5$	
		Analytical	Monte Carlo	Analytical	Monte Carlo
$\lambda_c = 0.01$	$\rho = 0.2$	19.0966	19.0908	27.5180	27.5294
	$c = 0.8$		95% CI:(19.01,19.17)		95% CI:(27.41,27.64)
	$\rho = 0.7$	18.3033	18.3268	27.2554	27.2452
	$c = -0.4$		95% CI:(18.27,18.38)		95% CI:(27.14,27.35)
	$\rho = -0.6$	20.5341	20.5772	28.0948	28.1324
	$c = 0.3$		95% CI:(20.47,20.68)		95% CI:(28.00,28.27)
	$\rho = -0.9$	21.0591	21.1134	28.3466	28.3486
	$c = -0.8$		95% CI:(21.00,21.23)		95% CI:(28.21,28.49)
$\lambda_c = 3.00$	$\rho = 0.2$	31.9122	31.8996	36.3562	36.3440
	$c = 0.8$		95% CI:(31.75,32.05)		95% CI:(36.16,36.52)
	$\rho = 0.7$	34.7039	34.6985	38.6616	38.7086
	$c = -0.4$		95% CI:(34.55,34.85)		95% CI:(38.52,38.89)
	$\rho = -0.6$	33.6269	33.6751	37.8362	37.8151
	$c = 0.3$		95% CI:(33.51,33.84)		95% CI:(37.62,38.01)
	$\rho = -0.9$	36.0160	35.9918	39.8927	39.9244
	$c = -0.8$		95% CI:(35.81,36.17)		95% CI:(39.71,40.14)
Average CPU Time Per Price		0.3 Sec	26 Sec	0.5 Sec	37 Sec

Table 2: Numerical results for the exchange option. The parameters used are $S_1(0) = 70$, $S_2(0) = 52$, $T = 0.25$, $r = 0.05$, $\sigma_1 = 0.3$, $\sigma_2 = 0.5$, $m_1 = -0.2$, $m_2 = 0.4$, $v_1 = 0.5$, $v_2 = 0.3$, $m_{1,c} = -0.5$,$m_{2,c} = 0.1$, $v_{1,c} = 0.4$, $v_{2,c} = 0.6$, $k = 0$. For the Monte Carlo results, 100,000 simulation runs are used. To reduce the discretization error in simulation, we use 2000 and 4000 discretization grids and then use Richardson's extrapolation for the square root convergence rate. As a comparison, for the Brownian model (i.e. $\lambda_1 = \lambda_2 = \lambda_c = 0$) the option prices are 19.0394, 18.2244, 20.4801, and 20.9978 for $\rho = 0.2$, 0.7, -0.6, -0.9, respectively. To verify the accuracy of the analytical solution, we compute the case of $\lambda_1 = \lambda_2 = 0$ twice by using both the Laplace inversion and the Υ function; both give the same results up to four decimal points.

Numerical results are given in Table 2 with different pairs of the dependence parameters c and ρ, and different common jump rates λ_c. All the computations are done by a Pentium 750 PC. In the table we illustrate two cases: (i) $\lambda_1 = \lambda_2 = 0$. (ii) $\lambda_1 = \lambda_2 = 0$. Case (i) is interesting because we can check the accuracy of the Laplace inversion (as we can also compute the prices via the Hh functions), and check the prices under the jump diffusion model against those under the pure Brownian model when the jump rates are very small. As it can been seen from Table 2, the analytical prices via the Laplace inversion can be obtained within a fractional second with good accuracy. To verify the accuracy of the Laplace inversion, we compute the case of $\lambda_1 = \lambda_2 = 0$ twice by using both the Laplace inversion and the Υ function; both give the same results up to four decimal points.

In Table 2 the option prices increase as the Poisson rate increases. Everything else being equal, negative correlations either in the jump part or the diffusion tend to lead to higher option prices, which is consistent with the fact that the payoff of an exchange option is an increasing function of the difference of the two asset prices.

A Appendix. Risk Neutral Pricing

Due to the presence of jumps, the market is incomplete and there is no unique state-price density (pricing kernel) to choose from to price options. In our case of the asymmetric Laplace jump diffusion with constant risk-free rate r, the general state-price density suggested in Duffie et al. (2001) becomes

$$Z_t = e^{-rt} \exp(\alpha(t) + \delta(t) \cdot X_t), \tag{37}$$

where $\delta(t) = (\delta_1(t), \delta_2(t))'$ and $\alpha(t)$ satisfy a set of ordinary differential equations

$$\begin{aligned} \delta_1(t) &= 0,\ \delta_2(t) = 0,\ \delta_1(T) = u_1,\ \delta_2(T) = u_2, \\ \dot{\alpha}(t) &= r - \mu \cdot \delta(t) - \frac{1}{2}\delta(t)\Sigma\delta(t)^T - \lambda(\varphi(\delta(t)) - 1),\ \alpha(T) = 0, \end{aligned}$$

with $\varphi(\delta(t)) = E[\exp\{\delta_1(t)Y^{(1)} + \delta_2(t)Y^{(2)}\}]$. The solution is given by

$$\begin{aligned} \delta_1(t) &= \delta_1,\ \delta_2(t) = \delta_2, \\ \alpha(t) &= -(T-t)\{r - (\mu_1 u_1 + \mu_2 u_2) - \frac{1}{2}(u_1, u_2)\Sigma(u_1, u_2)' - \lambda(E[e^{u_1 Y^{(1)} + u_2 Y^{(2)}} - 1])\}, \end{aligned}$$

If the parameters δ_1 and δ_2 are set so that $\mathsf{E}[Z_t]$ exists, then it is easy to see that Z_t is a martingale under P with $\mathsf{E}[Z_t] = 1$. Using the martingale property of Z_t, we can define a new probability measure P^*,

$$d\mathsf{P}^*/d\mathsf{P} = Z_t/Z_0.$$

For this particular choice of the state-price density, a European option with a payoff $g(T)$ at time T has a price

$$\mathsf{E}^*[g(T)|\mathcal{F}_t] = \frac{1}{Z_t}\mathsf{E}[g(T)Z_T|\mathcal{F}_t],$$

at time t. The next result shows that the measure transform from P to P^* preserves the distribution property of the asymmetric Laplace distribution.

Lemma A.1 *Under P^*, X_t in (14) becomes a new asymmetric Laplace jump diffusion process*

$$X_t = \mu^* t + \sigma W^*(t) + \sum_{i=1}^{N(t)} Y_i, \tag{38}$$

where $W^*(t) = W(t) - (\sigma^T\delta)t$ is the new standard Brownian motion in R^2, $\mu^* = \mu + \Sigma\delta$, and $N(t)$ is a new Poisson process with jump rate $\lambda^* = \lambda_c^* + \lambda_1^* + \lambda_2^*$. The new jump size Y still has the same form of the distribution

$$Y \sim \begin{cases} \mathcal{AL}_2(\ m_c^*,\ J^*), & w.p.\ \ \lambda_c^*/\lambda^* \\ (\mathcal{AL}_1(m_1^*,(v_1^*)^2),0)', & w.p.\ \ \lambda_1^*/\lambda^*, \\ (0,\mathcal{AL}_1(m_2^*,(v_2^*)^2))', & w.p.\ \ \lambda_2^*/\lambda^*, \end{cases}$$

with new parameters being

$$\begin{aligned} m_c^* &= (m_c + J\delta)\varphi_c(\delta), \quad J_c^* = J_c\varphi_c(\delta(t,T,b)),\ \lambda_c^* = \lambda_c\varphi_c(\delta), \\ m_k^* &= (m_k + \delta_k v_k^2)\varphi_k(\delta_k), \quad v_k^* = v_k\sqrt{\varphi_k(\delta_k)},\ \lambda_k^* = \lambda_k\varphi_k(\delta_k), \quad k = 1,2. \end{aligned}$$

Note that $\varphi_c(\cdot)$ and $\varphi_k(\cdot)$, $k = 1,2$, are given in (13). Furthermore, let $S_k(t) = S_k(0)e^{X_t^{(k)}}$ be the asset price. Then with $m_k^* + v_k^{*2}/2 < 1$ and $m_{k,c}^* + v_{k,c}^{*2}/2 < 1$ we have

$$S_k(t) = e^{-r(T-t)}\mathsf{E}^*[S_k(T)|\mathcal{F}_t], \qquad k = 1,2.$$

with the drift of $S_k(t)$ under P^* being

$$\mu_k^* = r - \frac{1}{2}\sigma_k^2 - \lambda^*\zeta_k^*,\ \zeta_k^* = \mathsf{E}^*(e^{Y^{(k)}}) - 1 = \frac{\lambda_k^*/\lambda^*}{1 - m_k^* - v_k^{*2}/2} + \frac{\lambda_c^*/\lambda^*}{1 - m_{k,c}^* - v_{k,c}^{*2}/2} + \frac{\lambda^* - \lambda_k^* - \lambda_c^*}{\lambda^*} - 1. \tag{39}$$

Thus, under the risk neutral measure the asset prices still follow a two-dimensional asymmetric Laplace jump-diffusion.

Proof. The result follows immediately from Lemma 2.2. □

B Appendix: Proof of Proposition 3.3

Proof. After plugging in the form of u, we have $\forall x_2 < b$,

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= -A_1\theta e^{-\theta x_1 - \beta_1(b-x_2)} - B_1\theta e^{-\theta x_1 - \beta_2(b-x_2)} - C_1\theta e^{-\theta x_1 - \beta_3(b-x_2)}, \\ \frac{\partial u}{\partial x_2} &= A_1\beta_1 e^{-\theta x_1 - \beta_1(b-x_2)} + B_1\beta_2 e^{-\theta x_1 - \beta_2(b-x_2)} + C_1\beta_3 e^{-\theta x_1 - \beta_3(b-x_2)}, \\ \frac{\partial^2 u}{\partial x_1^2} &= A_1\theta^2 e^{-\theta x_1 - \beta_1(b-x_2)} + B_1\theta^2 e^{-\theta x_1 - \beta_2(b-x_2)} + C_1\theta^2 e^{-\theta x_1 - \beta_3(b-x_2)}, \\ \frac{\partial^2 u}{\partial x_2^2} &= A_1\beta_1^2 e^{-\theta x_1 - \beta_1(b-x_2)} + B_1\beta_2^2 e^{-\theta x_1 - \beta_2(b-x_2)} + C_1\beta_3^2 e^{-\theta x_1 - \beta_3(b-x_2)}, \\ \frac{\partial^2 u}{\partial x_1 \partial x_2} &= -A_1\theta\beta_1 e^{-\theta x_1 - \beta_1(b-x_2)} - B_1\theta\beta_2 e^{-\theta x_1 - \beta_2(b-x_2)} - C_1\theta\beta_3 e^{-\theta x_1 - \beta_3(b-x_2)}. \end{aligned} \tag{40}$$

For $x_2 < b$, since $u(x_1+y_1,x_2) = e^{-\theta_1 y} u(x_1,x_2)$ we get

$$
\begin{aligned}
& \int_{-\infty}^{\infty} [u(x_1+y_1,x_2) - u(x_1,x_2)] f_{Y^{(1)}}(y_1) dy_1 \\
= \quad & u(x_1,x_2) \mathsf{E}[e^{-\theta Y^{(1)}} - 1] = u(x_1,x_2) \left(\frac{1}{1+\theta m_1 - v_1^2\theta^2/2} - 1 \right). \qquad (41)
\end{aligned}
$$

The other one-dimensional integral inside the generator $\mathcal{L}$ is

$$
\begin{aligned}
& \int_{-\infty}^{\infty} [u(x_1,x_2+y_2) - u(x_1,x_2)] f_{Y^{(2)}}(y_2) dy_2 \\
= \quad & \int_{-\infty}^{b-x_2} [A_1 e^{-\theta x_1 - \beta_1(b-x_2-y_2)} + B_1 e^{-\theta x_1 - \beta_2(b-x_2-y_2)} + C_1 e^{-\theta x_1 - \beta_3(b-x_2-y_2)}] f_{Y^{(2)}}(y_2) dy_2 \\
& + \int_{b-x_2}^{\infty} e^{-\theta x_1} f_{Y^{(2)}}(y_2) dy_2 - u(x_1,x_2).
\end{aligned}
$$

Lemma 2.1 implies that $Y^{(2)}$ actually has a double exponential density

$$
f_{Y^{(2)}}(y) = p^{(2)} \cdot \eta_1^{(2)} e^{-\eta_1^{(2)} y} 1_{\{y \geq 0\}} + q^{(2)} \cdot \eta_2^{(2)} e^{\eta_2^{(2)} y} 1_{\{y<0\}}, p^{(2)} > 0, q^{(2)} > 0, p^{(2)} + q^{(2)} = 1,
$$

where the parameters are given by

$$
\eta_1^{(2)} = \frac{2}{\sqrt{m_2^2 + 2v_2^2} + m_2}, \eta_2^{(2)} = \frac{2}{\sqrt{m_2^2+2v_2^2} - m_2},\ p^{(2)} = \frac{\sqrt{m_2^2+2v_2^2} + m_2}{2\sqrt{m_2^2+2v_2^2}}.
$$

Therefore,

$$
\begin{aligned}
\int_{-\infty}^{b-x_2} e^{\beta y_2} f_{Y^{(2)}}(y_2) dy_2 &= \int_{-\infty}^{0} e^{\beta y_2} q^{(2)} \eta_2^{(2)} e^{\eta_2^{(2)} y_2} dy_2 + \int_{0}^{b-x_2} e^{\beta y_2} p^{(2)} \eta_1^{(2)} e^{-\eta_1^{(2)} y_2} dy_2 \\
&= \frac{q^{(2)}\eta_2^{(2)}}{\beta + \eta_2^{(2)}} + \frac{p^{(2)}\eta_1^{(2)}}{\eta_1^{(2)} - \beta} \left(1 - e^{-(\eta_1^{(2)} - \beta)(b-x_2)} \right) \\
&= \frac{1}{1 - m_2\beta - \beta^2 v_2^2/2} + \frac{e^{-(\eta_1^{(2)}-\beta)(b-x_2)}}{\sqrt{m_2^2+2v_2^2}(\beta - \eta_1^{(2)})},
\end{aligned}
$$

$$
\int_{b-x_2}^{\infty} f_{Y^{(2)}}(y_2) dy_2 = \int_{b-x_2}^{\infty} p^{(2)} \eta_1^{(2)} e^{-\eta_1^{(2)} y_2} dy_2 = p^{(2)} e^{-\eta_1^{(2)}(b-x_2)} = \frac{e^{-\eta_1^{(2)}(b-x_2)}}{\eta_1^{(2)}\sqrt{m_2^2+2v_2^2}}.
$$

Note that $\eta_1^{(2)} = \beta_2^h$, where β_2^h is defined in (22). Hence,

$$
\begin{aligned}
\int_{-\infty}^{\infty} [u(x_1,x_2+y_2) - u(x_1,x_2)] f_{Y^{(2)}}(y_2) dy_2 &= \frac{e^{-\theta x_1 - \beta_2^h(b-x_2)}}{\sqrt{m_2^2+2v_2^2}} \left(\frac{A_1}{\beta_1 - \beta_2^h} + \frac{B_1}{\beta_2 - \beta_2^h} + \frac{C_1}{\beta_3 - \beta_2^h} + \frac{1}{\beta_2^h} \right) \\
+ \frac{A_1 e^{-\theta x_1 - \beta_1(b-x_2)}}{1 - \beta_1 m_2 - \frac{1}{2}v_2^2\beta_1^2} + \frac{B_1 e^{-\theta x_1 - \beta_2(b-x_2)}}{1 - \beta_2 m_2 - \frac{1}{2}v_2^2\beta_2^2} &+ \frac{C_1 e^{-\theta x_1 - \beta_3(b-x_2)}}{1 - \beta_3 m_2 - \frac{1}{2}v_2^2\beta_3^2} - u(x_1,x_2). \qquad (42)
\end{aligned}
$$

The two-dimensional integral inside $\mathcal{L}$ becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[u(x_1+y_1,x_2+y_2)-u(x_1,x_2)]f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_1dy_2\\
=\;&\int_{-\infty}^{\infty}dy_1\int_{-\infty}^{b-x_2}u(x_1+y_1,x_2+y_2)f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_2\\
&+\int_{-\infty}^{\infty}dy_1\int_{b-x_2}^{\infty}u(x_1+y_1,x_2+y_2)f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_2-u(x_1,x_2)\\
=\;&A_1e^{-\theta x_1-\beta_1(b-x_2)}\int_{-\infty}^{\infty}dy_1\int_{-\infty}^{b-x_2}e^{-\theta y_1+\beta_1y_2}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_2\\
&+B_1e^{-\theta x_1-\beta_2(b-x_2)}\int_{-\infty}^{\infty}dy_1\int_{-\infty}^{b-x_2}e^{-\theta y_1+\beta_2y_2}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_2\\
&+C_1e^{-\theta x_1-\beta_3(b-x_2)}\int_{-\infty}^{\infty}dy_1\int_{-\infty}^{b-x_2}e^{-\theta y_1+\beta_3y_2}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_2\\
&+e^{-\theta x_1}\int_{-\infty}^{\infty}dy_1\int_{b-x_2}^{\infty}e^{-\theta y_1}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_2-u(x_1,x_2).
\end{aligned}$$

To compute this, we first consider integral of type $\int_{-\infty}^{\infty}dy_1\int_{-\infty}^{b-x_2}e^{-\theta y_1+\beta y_2}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_2$. The density function of the bivariate asymmetric Laplace distribution is given in Kotz et al. (2001, p.241):

$$f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)=\frac{e^{Ay_1+By_2}}{\pi v_1v_2\sqrt{1-c^2}}\cdot\frac{1}{2}\int_0^{\infty}\frac{1}{t}\exp\left(-t-\frac{K^2(y_1^2v_{2,c}/v_{1,c}-2cy_1y_2+y_2^2v_{1,c}/v_{2,c})}{4t}\right)dt$$

$$\begin{aligned}
A&=\frac{m_{1,c}v_{2,c}/v_{1,c}-m_{2,c}c}{v_{1,c}v_{2,c}(1-c^2)},\quad B=\frac{m_{2,c}v_{1,c}/v_{2,c}-m_{1,c}c}{v_{1,c}v_{2,c}(1-c^2)},\\
K&=\frac{\{2v_{1,c}v_{2,c}(1-c^2)+m_{1,c}^2v_{2,c}/v_{1,c}-2cm_{1,c}m_{2,c}+m_{2,c}^2v_{1,c}/v_{2,c}\}^{1/2}}{v_{1,c}v_{2,c}(1-c^2)}.
\end{aligned}$$

Thus, completing squares yields

$$\begin{aligned}
&\int_{-\infty}^{\infty}e^{-\theta y_1}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_1\\
=\;&\int_0^{\infty}\frac{e^{-t}}{t}\frac{\exp\left(By_2-\frac{K^2v_{1,c}}{4tv_{2,c}}y_2^2\right)}{2\pi v_{1,c}v_{2,c}\sqrt{1-c^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{K^2v_{2,c}}{4tv_{1,c}}y_1^2+(\frac{cy_2K^2}{2t}+A-\theta)y_1\right\}dy_1dt\\
=\;&\int_0^{\infty}\frac{e^{-t}}{t}\frac{\exp\left(By_2-\frac{K^2v_{1,c}}{4tv_{2,c}}y_2^2+\frac{tv_{1,c}}{K^2v_{2,c}}(\frac{cy_2K^2}{2t}+A-\theta)^2\right)}{2\pi v_{1,c}v_{2,c}\sqrt{1-c^2}}\cdot\sqrt{\frac{4tv_{1,c}\pi}{K^2v_{2,c}}}dt\\
=\;&\frac{\exp\left(By_2+\frac{cv_{1,c}}{v_{2,c}}(A-\theta)y_2\right)}{\sqrt{\pi}v_{1,c}v_{2,c}\sqrt{1-c^2}}\sqrt{\frac{v_{1,c}}{K^2v_{2,c}}}\int_0^{\infty}\frac{1}{\sqrt{t}}\exp\left(-\frac{K^2v_{2,c}-v_{1,c}(A-\theta)^2}{K^2v_{2,c}}t-\frac{K^2v_{1,c}y_2^2(1-c^2)}{4v_{2,c}}\frac{1}{t}\right)dt.
\end{aligned}$$

Using the fact that

$$\int_0^\infty \frac{1}{\sqrt{t}} \exp\left\{-bt - \frac{a}{2t}\right\} dt = \sqrt{\frac{1}{b}} \int_0^\infty \frac{1}{\sqrt{s}} \exp\left\{-s - \frac{ab}{4s}\right\} ds = e^{-\sqrt{ab}} \sqrt{\frac{\pi}{b}},\ a, b > 0,$$

which follows from the definition of the modified Bessel function of the third kind (Bateman, 1953, p. 5, Bateman, 1954, p. 146), we have

$$\int_{-\infty}^\infty e^{-\theta y_1} f^c_{(Y^{(1)},Y^{(2)})}(y_1, y_2) dy_1 = \frac{\exp\left(By_2 + \frac{cv_{1,c}}{v_{2,c}}(A-\theta)y_2 - \frac{|y_2|}{v_{2,c}}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}\right)}{v_{2,c}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}},$$

because

$$K^2 v_{2,c} - v_{1,c}(A-\theta)^2 = \frac{(\beta_2^g - \beta_1^g)^2 v_{2,c}^2}{v_{1,c}(1-c^2)} > 0,$$

where β_1^g, β_2^g are given in (21). Therefore,

$$\begin{aligned} & \int_{-\infty}^{b-x_2} e^{\beta y_2} dy_2 \int_{-\infty}^\infty e^{-\theta y_1} f^c_{(Y^{(1)},Y^{(2)})}(y_1, y_2) dy_1 \\ = & \int_{-\infty}^0 \frac{\exp\left((\beta + B + \frac{cv_{1,c}}{v_{2,c}}(A-\theta))y_2 + \frac{y_2}{v_{2,c}}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}\right)}{v_{2,c}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}} dy_2 \\ & + \int_0^{b-x_2} \frac{\exp\left((\beta + B + \frac{cv_{1,c}}{v_{2,c}}(A-\theta))y_2 - \frac{y_2}{v_{2,c}}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}\right)}{v_{2,c}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}} dy_2. \end{aligned}$$

Plugging in the values of K, A, and B, and denoting

$$\begin{aligned} D &= v_{2,c}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]} \\ &= \sqrt{m_{2,c}^2 - 2cv_{1,c}v_{2,c}m_{2,c}\theta + 2v_{2,c}^2 + 2v_{2,c}^2 m_{1,c}\theta - \theta^2 v_{1,c}^2 v_{2,c}^2(1-c^2)} = (\beta_2^g - \beta_1^g)v_{2,c}^2/2, \\ \xi &= B + \frac{cv_{1,c}}{v_{2,c}}(A-\theta) - \frac{1}{v_{2,c}}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]} \\ &= \frac{m_{2,c}}{v_{2,c}^2} - \frac{cv_{1,c}\theta}{v_{2,c}} - \frac{D}{v_{2,c}^2} = -\beta_2^g, \end{aligned}$$

we get

$$\begin{aligned} & \int_{-\infty}^0 \frac{\exp\left((\beta + B + \frac{cv_{1,c}}{v_{2,c}}(A-\theta))y_2 + \frac{y_2}{v_{2,c}}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}\right)}{v_{2,c}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}} dy_2 \\ & = \frac{1}{D}\int_{-\infty}^0 e^{(\beta+\xi+2D/v_{2,c}^2)y_2} dy_2 = \frac{1}{D(\beta + \xi + 2D/v_{2,c}^2)}, \end{aligned}$$

$$\begin{aligned} & \int_0^{b-x_2} \frac{\exp\left((\beta + B + \frac{cv_{1,c}}{v_{2,c}}(A-\theta))y_2 - \frac{y_2}{v_{2,c}}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}\right)}{v_{2,c}\sqrt{v_{1,c}(1-c^2)[K^2 v_{2,c} - v_{1,c}(A-\theta)^2]}} dy_2 \\ & = \frac{1}{D}\int_0^{b-x_2} e^{(\beta+\xi)y_2} dy_2 = \frac{e^{(\beta+\xi)(b-x_2)} - 1}{D(\beta+\xi)}. \end{aligned}$$

Note that

$$\frac{1}{D}\left(\frac{1}{\beta+\xi+2D/v_{2,c}^2}-\frac{1}{\beta+\xi}\right)=1+\theta m_{1,c}-\beta m_{2,c}-\frac{1}{2}v_{1,c}^2\theta^2-\frac{1}{2}v_{2,c}^2\beta^2+cv_{1,c}v_{2,c}\theta\beta=1/g(\beta),$$

where $g(\cdot)$ is given in (23). Hence, we can compute the two-dimensional integral as

$$\begin{aligned}&\int_{-\infty}^{b-x_2}e^{\beta y_2}dy_2\int_{-\infty}^{\infty}e^{-\theta y_1}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_1\\ =\ &\frac{1}{D(\beta+\xi+2D/v_{2,c}^2)}+\frac{e^{(\beta+\xi)(b-x_2)}-1}{D(\beta+\xi)}\\ =\ &\frac{e^{(\beta+\xi)(b-x_2)}}{(\beta+\xi)D}+1/g(\beta)=\frac{e^{(\beta-\beta_2^g)(b-x_2)}}{(\beta-\beta_2^g)(\beta_2^g-\beta_1^g)v_{2,c}^2/2}+1/g(\beta),\end{aligned}$$

which can also be used to calculate

$$\begin{aligned}&\int_{-\infty}^{\infty}dy_1\int_{b-x_2}^{\infty}e^{-\theta y_1}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_2\\ =\ &\int_{-\infty}^{\infty}dy_2\int_{-\infty}^{\infty}e^{-\theta y_1}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_1-\int_{-\infty}^{b-x_2}dy_2\int_{-\infty}^{\infty}e^{-\theta y_1}f^c_{(Y^{(1)},Y^{(2)})}(y_1,y_2)dy_1\\ =\ &\frac{1}{1+\theta m_{1,c}-\theta^2 v_{1,c}^2/2}-\left(\frac{e^{-\beta_2^g(b-x_2)}}{-\beta_2^g(\beta_2^g-\beta_1^g)v_{2,c}^2/2}+1/g(0)\right)=\frac{e^{-\beta_2^g(b-x_2)}}{\beta_2^g(\beta_2^g-\beta_1^g)v_{2,c}^2/2}.\end{aligned}$$

In summary,

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[u(x_1+y_1,x_2+y_2)-u(x_1,x_2)]f^c_{(Y^{(1)},Y^{(2)})}(y_2)dy_1dy_2\\ =\ &\frac{e^{-\theta x_1-\beta_2^h(b-x_2)}}{(\beta_2^g-\beta_1^g)v_{2,c}^2/2}\left(\frac{A_1}{\beta_1-\beta_2^g}+\frac{B_1}{\beta_2-\beta_2^g}+\frac{C_1}{\beta_3-\beta_2^g}+\frac{1}{\beta_2^g}\right)\\ &+A_1e^{-\theta x_1-\beta_1(b-x_2)}g(\beta_1)+B_1e^{-\theta x_1-\beta_2(b-x_2)}g(\beta_2)+C_1e^{-\theta x_1-\beta_3(b-x_2)}g(\beta_3)-u(x_1,x_2).\end{aligned}\tag{43}$$

The result follows by combining (40), (41), (42), and (43). □

C First Passage Times and Barrier Options in the Case of $\lambda_1 = \lambda_2 = 0$

In this case, the process in (16) can be simplified into the case that $(Y^{(1)},Y^{(2)})'\sim\mathcal{AL}_2(m_c,J_c)$. Thus, $\mathsf{E}\left[e^{-\theta X_t^{(1)}+\beta X_t^{(2)}}\right]=\exp\{M(\theta,\beta)t\}$, where the function $M(\cdot,\cdot)$ is defined as

$$\begin{aligned}M(x,y)\ =\ &-\mu_1x+\mu_2y+\frac{1}{2}\sigma_1^2x^2+\frac{1}{2}\sigma_2^2y^2-\rho\sigma_1\sigma_2xy\\ &+\frac{\lambda}{1+xm_{1,c}-ym_{2,c}-\frac{1}{2}v_{1,c}^2x^2-\frac{1}{2}v_{2,c}^2y^2+cv_{1,c}v_{2,c}xy}-\lambda\end{aligned}\tag{44}$$

Lemma C.1 *There exists $\theta_0 > 0$, such that for all $\theta \in (0, \theta_0)$, the equation $M(\theta, y) = \alpha$, $\forall \alpha > 0$, has exactly four roots: $\beta_1, \beta_2, \beta_3, \beta_4$, where $0 < \beta_{1,\alpha} < \beta_2^g < \beta_{2,\alpha} < \infty$, $-\infty < \beta_{3,\alpha} < \beta_1^g < \beta_{4,\alpha} < 0$. Here β_1^g and β_2^g are defined as in (21).*

Proof. The proof is similar to that of Proposition 3.1. The only difference is that in Proposition 3.1, $M_e(\theta, \cdot)$ has 4 discontinuous points, β_1^h, β_2^h, β_1^g and β_2^g. While here $M(\theta, \cdot)$ has only 2 discontinuous points β_1^g and β_2^g. Hence, instead of dividing the real line into 6 parts, we divide it into 4 parts $(-\infty, \beta_1^g)$, $(\beta_1^g, 0)$, $(0, \beta_2^g)$, (β_2^g, ∞). The rest of argument is the same as in Lemma 3.1. □

Theorem C.2 *Suppose we have two jump diffusion processes defined as in (16) with $\lambda_1 = \lambda_2 = 0$. For some small enough $\theta > 0$, let β_1 and β_2 be the only two positive roots for the equation $M(\theta, y) = \alpha$, $\forall \alpha > 0$, where $0 < \beta_1 < \beta_2^g < \beta_2 < \infty$. Then*

$$\mathsf{E}[e^{-\alpha\tau_b} e^{-\theta X_{\tau_b}^{(1)}} 1_{\{\tau_b < \infty\}}] = \frac{\beta_2^g \beta_2 - \beta_1 \beta_2}{\beta_2^g \beta_2 - \beta_2^g \beta_1} e^{-\beta_1 b} + \frac{\beta_1 \beta_2 - \beta_2^g \beta_1}{\beta_2^g \beta_2 - \beta_2^g \beta_1} e^{-\beta_2 b},$$

with β_2^g defined in (21).

Proof. For any fixed level b, we define function u to be:

$$u(x_1, x_2) = \begin{cases} e^{-\theta x_1}, & x_2 \geq b \\ A_1 e^{-\theta x_1 - \beta_1(b - x_2)} + B_1 e^{-\theta x_1 - \beta_2(b - x_2)}, & x_2 < b \end{cases}$$

The infinitesimal generator of the two jump processes is simplified into

$$\begin{aligned} \mathcal{L}u &= \mu_1 \frac{\partial u}{\partial x_1} + \mu_2 \frac{\partial u}{\partial x_2} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 u}{\partial x_2^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ &\quad + \lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [u(x_1 + y_1, x_2 + y_2) - u(x_1, x_2)] f_{(Y^{(1)}, Y^{(2)})}(y_1, y_2) dy_1 dy_2, \end{aligned}$$

for all continuous twice differentiable function $u(x_1, x_2)$. Similar to the proof of Proposition 3.3, we can calculate the partial derivatives of $u(x_1, x_2)$ and the two-dimensional integral. After some algebra, we get that for all $x_2 < b$, $-\alpha u + \mathcal{L}u$ is equal to

$$A_1 e^{-\theta x_1 - \beta_1(b - x_2)} f(\beta_1) + B_1 e^{-\theta x_1 - \beta_2(b - x_2)} f(\beta_2) + \frac{\lambda e^{-\theta x_1 - \beta(b - x_2)}}{(\beta_2^g - \beta_1^g) v_2^2 / 2} \left(\frac{A_1}{\beta_1 - \beta_2^g} + \frac{B_1}{\beta_2 - \beta_2^g} + \frac{1}{\beta_2^g} \right),$$

where

$$\begin{aligned} f(\beta) &= -\mu_1\theta + \mu_2\beta + \frac{1}{2}\sigma_1^2\theta^2 + \frac{1}{2}\sigma_2^2\beta^2 - \rho\sigma_1\sigma_2\theta\beta \\ &+ \frac{\lambda}{1 + \theta m_{1,c} - \beta m_{2,c} - \frac{1}{2}v_{1,c}^2\theta^2 - \frac{1}{2}v_{2,c}^2\beta^2 + c v_{1,c} v_{2,c}\theta\beta} - \lambda - \alpha. \end{aligned}$$

Then applying Lemma C.1, we know that there exist $\beta_1 > 0, \beta_2 > 0$ such that $M(\theta, \beta_1) = M(\theta, \beta_2) = \alpha$, i.e. $f(\beta_1) = f(\beta_2) = 0$, by solving

$$\frac{A_1}{\beta_1 - \beta_2^g} + \frac{B_1}{\beta_2 - \beta_2^g} + \frac{1}{\beta_2^g} = 0, \quad A_1 + B_1 = 1,$$

we have

$$A_1 = \frac{\beta_2^g \beta_2 - \beta_1 \beta_2}{\beta_2^g \beta_2 - \beta_2^g \beta_1}, \qquad B_1 = 1 - A_1.$$

Thus, we have

$$-\alpha u + \mathcal{L}u = 0, \qquad \forall (x_1, x_2),\ x_2 < b.$$

The rest of argument is the same as in proof of Theorem 3.2. □

Theorem C.3 *For any two jump diffusion processes defined as in (16) with $\lambda_1 = \lambda_2 = 0$, denote*

$$\Omega(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho, \lambda, m_{1,c}, m_{2,c}, v_{1,c}, v_{2,c}, c; a, b, t) = \mathsf{P}(X_t^{(1)} \le a, \tau_b \le t).$$

Then we have that

$$\int_{-\infty}^{\infty} e^{-\theta a} \int_0^{\infty} e^{-\alpha t} \mathsf{P}(X_t^{(1)} \le a, \tau_b \le t) dt da = \frac{1}{\theta(\alpha - M(\theta, 0))} \left(\frac{\beta_2^g \beta_2 - \beta_1 \beta_2}{\beta_2^g \beta_2 - \beta_2^g \beta_1} e^{-\beta_1 b} + \frac{\beta_1 \beta_2 - \beta_2^g \beta_1}{\beta_2^g \beta_2 - \beta_2^g \beta_1} e^{-\beta_2 b} \right),$$

where $M(\cdot, \cdot)$ is given in (44), β_2^g in (21), and β_1, β_2 in Theorem C.2.

Proof. It's the same argument as for Theorem 3.4, except replacing $M_e(\cdot, \cdot)$ by $M(\cdot, \cdot)$, and $E[e^{-\alpha \tau_b} e^{-\theta X_{\tau_b}^{(1)}} 1_{\{\tau_b < \infty\}}]$ is given by Theorem C.2. □

Theorem C.4 *Suppose we have two underlying assets as in (9) with $\lambda_1 = \lambda_2 = 0$. Then the price of up-and-in put barrier option on these two assets is*

$$\begin{aligned} UIP \;=\; & K e^{-rT} \Omega(r - \frac{1}{2}\sigma_1^2 - \lambda \zeta_1, r - \frac{1}{2}\sigma_2^2 - \lambda \zeta_2, \sigma_1, \sigma_2, \lambda, m_{1,c}, m_{2,c}, v_{1,c}, v_{2,c}, c; \\ & \log(K/S_1(0)), \log(H/S_2(0)), T) \\ & - S_1(0) \Omega(r + \frac{1}{2}\sigma_1^2 - \lambda \zeta_1, r - \frac{1}{2}\sigma_2^2 + \rho \sigma_1 \sigma_2 - \lambda \zeta_2, \sigma_1, \sigma_2, \widehat{\lambda}, \widehat{m}_{1,c}, \widehat{m}_{2,c}, \widehat{v}_{1,c}, \widehat{v}_{2,c}, \widehat{c}; \\ & \log(K/S_1(0)), \log(H/S_2(0)), T), \end{aligned}$$

where $\Omega(\cdot)$ is from Theorem C.3, $\widehat{m}_1, \widehat{m}_2, \widehat{v}_1, \widehat{v}_2, \widehat{c}$ and $\widehat{\lambda}$ are the same as in Theorem 4.1 except that setting $\lambda_1 = \lambda_2 = 0$.

Proof. The proof is essentially the same as the proof of Theorem 4.1. The only difference is that we set $\lambda_1 = \lambda_2 = 0$, so the function $\Omega_e(\cdot)$ is changed into $\Omega(\cdot)$. □

D Numerical Inversion of Laplace Transforms for the Barrier Options

There are several standard ways to invert Laplace transforms numerically. For example, one can use Gaver-Stehfest algorithm to invert Laplace transforms on the real domain, if high precision calculation is feasible (e.g. in the software package MATHEMATICA). One can also use Euler algorithm to invert Laplace transforms on the complex domain, which will not require high precision calculation. For a survey of various inversion methods, see Abate and Whitt (1992).

In this paper we shall use the two-sided Euler inversion algorithm in Petrella (2004). To do it we have to check that the error bounds in Petrella (2004) are satisfied. For any function $f(\cdot,\cdot)$ in $R^+ \times R$ let its Laplace transform be $\widehat{f}(s_1,s_2) = \int_0^\infty \int_{-\infty}^\infty e^{-s_1 t_1 - s_2 t_2} f(t_1,t_2) dt_1 dt_2$. Suppose we want to invert at a point (t_1^0, t_2^0). If

$$f((2j_1+1)t_1^0, (2j_2+1)t_2^0) \le \left\{ \begin{array}{cc} C^+ & \text{for} \quad j_1 \ge 0, j_2 \ge 0 \\ C^- e^{\nu_1 j_1 t_1^0 + \nu_2 j_2 t_2^0} & \text{for} \quad j_1 < 0, j_2 \ge 0 \end{array} \right\} \tag{45}$$

with $\nu_1 > 0, \nu_2 > 0$ and $t_1^0 \in (\frac{A_1}{\nu_1}, \infty), t_2^0 \in (0, \frac{A_1}{\nu_2})$, then Petrella (2004) gives the following bound for the inversion error

$$e_d \le \frac{C^+(e^{-A_1} + e^{-A_2})}{(1-e^{-A_1})(1-e^{-A_2})} + \frac{C^- e^{-(\nu_1 t_1^0 - A_1)}}{(1-e^{-(\nu_1 t_1^0 - A_1)})(1-e^{-(A_2 - \nu_2 t_2^0)})}.$$

Hence the error can be made small by choose A_1 and A_2 large enough. In our case, we have $t_1^0 = a = \ln(K/S_1(0)), t_2^0 = T$ and $f(t_1^0, t_2^0) = f(a,T) = \mathsf{P}(X_T^{(1)} \le a, \tau_b \le T)$. Setting $C^+ = 1$, we have for any $j_1 \ge 0, j_2 \ge 0$,

$$f((2j_1+1)t_1^0, (2j_2+1)t_2^0) = \mathsf{P}(X_{(2j_2+1)T}^{(1)} \le (2j_1+1)a, \tau_b \le (2j_2+1)T) \le 1 = C^+.$$

For any $j_1 < 0, j_2 \ge 0$, we can use the Markov inequality

$$\begin{aligned} f((2j_1+1)t_1^0, (2j_2+1)t_2^0) &= \mathsf{P}(X_{(2j_2+1)T}^{(1)} \le (2j_1+1)a, \tau_b \le (2j_2+1)T) \\ &\le \mathsf{P}(X_{(2j_2+1)T}^{(1)} \le (2j_1+1)a) \\ &\le e^{\nu(2j_1+1)a} E[e^{-\nu X_{(2j_2+1)T}^{(1)}}] = e^{\nu(2j_1+1)a + M_e(\nu,0)(2j_2+1)T}, \end{aligned}$$

where $M_e(\cdot,\cdot)$ is defined in (18) and $\nu > 0$. With $\nu_1 = 2\nu, \nu_2 = 2M_e(\nu,0)$, and $C^- = e^{a+M_e(\nu,0)T}$, the condition in (45) is satisfied.

In our inversion, we generally choose $\nu \approx 1.0$. Note that ν cannot be too big because we have to make sure that $\mathsf{E}[e^{-\nu X_{(2j_2+1)T}^{(1)}}]$ is well defined. And then we can calculate ν_1, ν_2. Generally we set $A_1 \approx 20, A_2 \approx 20 + \nu_2 t_2^0$ so that we can get an accuracy about 10^{-2} for option prices.

In the calculation of the Laplace transform we need to find three positive roots of $M_e(\theta, y) = \alpha$ for some $\theta \in (0, \theta_0)$ and $\alpha > 0$. In the Euler inversion we have to track the three roots in the complex domain after we add imaginary parts to θ and α. It can be shown that the analytical extensions of the three positive roots in the real domain become three roots with positive real parts in the complex domain. There are no closed-form solutions of the roots, since $M_e(\theta, y) = \alpha$ is equivalent to a 6-degree polynomial of y. To find the three roots numerically, we first use the Newton-Raphson method to find two different roots y_1, y_2, by choosing two different starting points at $y_1^{(0)} = \beta_2^h - \epsilon$ and $y_2^{(0)} = \beta_2^g - \epsilon$, where ϵ is an arbitrarily small positive real number. Here β_2^h and β_2^g are defined as in Lemma 3.1. Then after finding y_1 and y_2 we can divide $g(y)$ by $(y-y_1)(y-y_2)$, which yields a 4-degree polynomial. Therefore, the other four roots can be obtained by using the closed-form solution for the roots of 4-degree polynomials as in Kou, Petrella, and Wang (2005).

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