Pricing and Hedging Gap Risk

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Examples of gap risk

Gap risk: risk associated to jumps or rapid moves in the underlying value

- CPPI portfolio breaking through the floor due to a sudden jump in the underlying value
- A sudden drop in the value of a collateral guaranteeing a liability (e.g. stock loan)
- Various leveraged products in the credit world (leveraged CLN's)
- Gap options: options whose pay-off depends on the number of jumps of the underlying

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Gap options

- Gap options are a new class of exotic options offering protection against rapid downside market moves (gaps).
- Many different names: gap note, crash note, gap option, daily cliquet, gap risk swap etc.
- Usually come in the form of a boosted monetary product whose notional is reduced in case of a gap event.
- The protection buyers are banks who want to get rid of gap risk associated to CPPI strategies; the protection sellers are hedge funds.

Example of a single name gap option

- The protection seller pays the notional amount to the protection buyer (bank) and receives Libor + spread monthly plus the notional at maturity if no gap event occurs.
- The gap event is defined as a downside move of over 10% in the DJ Euro Stoxx 50 index within 1 day.
- In case of gap, seller receives $N(1 - 10 * (0.9 - R))^+$, where $R = \frac{S_t}{S_{t-1}}$ is the index performance.

Pay-off of a gap option

- The probability that a gap option pays off is low … but definitely not zero.

STOXX50	CAC40	DAX	S&P500
-8.2%	-9.5%	-7.4%	-9.5%

Strongest negative return of market indices during the last year.

- $\Rightarrow$ Gap options cannot be evaluated with continuous models: for a 28% volatility (highest historical VIX level before Lehman), the probability of having a 10% gap on any one day during one year is 2×10^{-6}.

Formal definition and 1st formula

- Let α be the return level which triggers the gap and k^* the time of first gap: $k^* := \inf\{k : R_k \leq \alpha\}$ (where $R_k = \frac{S_k}{S_{k-1}}$).
- The gap option pays to its holder the amount $f(R_{k^*})$ at time k^*, if $k^* \leq N$ and nothing otherwise.
- Let $(R_k)_{k=1}^N$ be i.i.d. and denote the distribution of $\log R_1$ by $p(dx)$. Then the price of a gap option is

$$\begin{aligned} G_\Delta &= E\left[e^{-\Delta k^* r} f(R_{k^*}) 1_{k^* \leq N}\right] \\ &= e^{-r\Delta} \int_{-\infty}^{\beta} f(e^x) p(dx) \frac{1 - e^{-rT}\left(\int_\beta^\infty p(dx)\right)^N}{1 - e^{-r\Delta}\int_\beta^\infty p(dx)}, \end{aligned}$$

with $\beta := \log \alpha < 0$ and $\Delta = 1$ day.

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Numerical evaluation by Fourier transform

- Let ϕ_Δ be the characteristic function of p_Δ. Then,

$$\int_{-\infty}^{\beta} p_\Delta(dx) = F_\Delta(x) = F'(x) + \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\frac{\phi'(u)-\phi_\Delta(u)}{iu}du,$$

  where F' is the CDF and ϕ' the characteristic function of a Gaussian variable (needed only for regularization).
- Similarly

$$e^{-r\Delta}\int_{-\infty}^{\beta} f(e^x)p_\Delta(dx)$$

  is the price of an option with payoff $f(e^x)1_{x\leq\beta}$ and maturity Δ, which can be evaluated by Fourier inversion.
- When Δ is small, the corresponding integrals converge very slowly.

Lévy processes and jump diffusions

Suppose that S may be written

$$S_t = S_0 e^{X_t}$$

where X is a Lévy process (jump-diffusion)

$$X_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k$$

with N_t Poisson process with intensity λ and (Y_k) i.i.d. with density f. The product $\nu(dx) \equiv \lambda f(dx)$ is called the *Lévy measure*.

- For general Lévy processes, one may have $\nu(\mathbb{R}) = \infty$: the process has infinitely many small jumps.

Approximate pricing formula

- For all Lévy processes,

$$\int_{-\infty}^{\beta} g(x) p_\Delta(dx) \sim \Delta \int_{-\infty}^{\beta} g(x)\nu(dx),$$

as $\Delta \to 0$, where ν is the Lévy measure of X (jump intensity × jump size distribution).

$$G_\Delta \approx G_0 = \lim_{\Delta\to 0} G_\Delta = \int_{-\infty}^{\beta} f(e^x)\nu(dx) \frac{1 - e^{-rT - T\int_{-\infty}^{\beta}\nu(dx)}}{r + \int_{-\infty}^{\beta}\nu(dx)}.$$

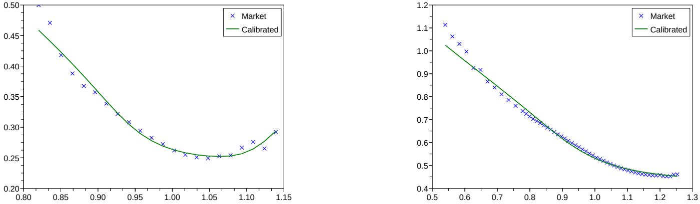
Numerical application: Kou's model

Kou's model: jump-diffusion model where the process X has a Gaussian component with volatility σ and a Lévy density

$$\nu(x) = \frac{\lambda(1-p)}{\eta_+} e^{-x/\eta_+} 1_{x>0} + \frac{\lambda p}{\eta_-} e^{-|x|/\eta_-} 1_{x<0}.$$

- Impossible to estimate the probability of a 10% gap from historical data: from 2002 to 2008, the strongest negative return was -8.2%.
- The probability of sharp downside moves can be extracted from short dated OTM put prices by calibrating an exponential Lévy model to market option quotes.

Calibration of Kou's model



Observed and calibrated implied volatilities of options on the DJ Euro Stoxx 50.
Left: 10-day options on July 7, 2008, calibrated parameters: $\sigma = 0.23$, $\lambda = 7.04$, $p = 0.985$, $\eta_- = 0.0414$.
Right: 12 day options on December 3, 2008, calibrated parameters: $\sigma = 0.39$, $\lambda = 10.02$, $p = 0.924$, $\eta_- = 0.104$.
Only points corresponding to non-zero transaction volumes shown.

Kou's model: gap option prices

- In Kou's model the approximate pricing formula is explicit:

$$\int_{-\infty}^{\beta} \nu(dx) = \lambda p e^{\beta/\eta_-}$$

and if we set $f(x) = (K - x)^+$ with $\log K \leq \beta$ then

$$\int_{-\infty}^{\beta} f(e^x)\nu(dx) = \frac{\lambda p \eta_-}{1+\eta_-} K^{1+1/\eta_-}.$$

- With the exact formula we obtained a price of 15.1%, and the approximate formula gives 14.3% (data from July 2008).
- For December 2008 data both formulas give 58%.

Hedging gap options

- The gap option is a zero-delta product and cannot be hedged with the underlying.
- The risk can be partially offset with low-delta instruments sensitive to extreme jumps, such as short-dated OTM puts.
- Since one-day OTM puts are not quoted, only approximate hedges (e.g. minimizing the L^2 error) can be constructed.

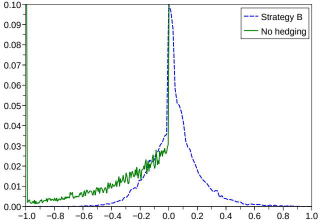
Hedging gap options

Let $P(t,S)$ be the price of the hedging option. Then the optimal quadratic hedge ratio is

$$\hat{\phi}_t = 1_{t \leq \tau} \frac{\int_{-\infty}^{\beta} \nu(dz) f(e^z) e^{-\lambda^*(T-t)} \{P(t, S_t e^z) - P(t, S_t)\}}{\sigma^2 S_t^2 \left(\frac{\partial P}{\partial S}\right)^2 + \int_{\mathbb{R}} \nu(dz) \{P(t, S_t e^z) - P(t, S_t)\}^2}.$$

- The optimal strategy requires continuous rebalancing, which is impossible in practice.
- In the numerical example we use the strategy: Buy $\hat{\phi}_0$ options at $t = 0$ and keep until maturity (1 week) *or until gap*.

Numerical example: Kou's model

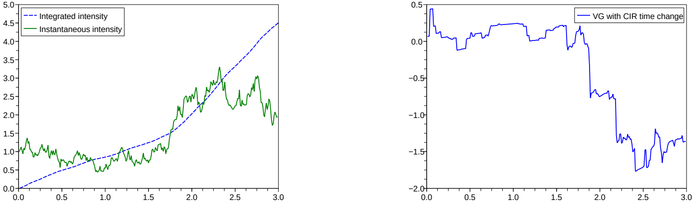


Histograms of P&L with and without hedging. In presence of hedging, the standard deviation is reduced from 4.7% to 2.3% and the 1-week VaR at the level of 99.9% is reduced from 0.85 to 0.23 (the parameters were those from July 2008).

Making the jump intensity stochastic

- The Lévy modeling assumes stationary returns and disregards the risk of future changes in jump intensity and jump clustering; the price of a gap option is constant and there is no mark to market risk.
- This risk can be incorporated by making the jump intensity stochastic, in a time-changed Lévy model (Carr et al. '03).
- In time-changed Lévy model (with integrated CIR time change) explicit pricing formulas for gap options are available.

A stochastic intensity model with jump clustering



Typical trajectory of the intensity process (left) and of the time-changed Lévy process (right) in the Variance Gamma model with CIR time change.

Multi-name gap options

- A basket of M (for example, 10) underlying stocks
- The gap event occurs if any one stock has a performance less than α (e.g., 80%) on one day
- Monthly payment of Libor $+$ spread
- Notional determined by the number N of gap events:

N	0	1	2	3	≥ 4
Notional	1	1	1	0.5	0

Pricing multi-name gap options

- Multi-dimensional exp-Lévy model $S_t^i = S_0^i e^{X_t^i}$, where $(X^1, \dots, X^M)$ is an M-dimensional Lévy process with Lévy measure ν (measure on $\mathbb{R}^M$ describing the intensity of simultaneous jumps in all components).
- Same approximation as before: gap event $\approx$ negative jump of size less than $\beta = \log(\alpha)$ in any of the components of X.
- The difficulty is to model simultaneous jumps in asset prices.

The tails of a Lévy measure

The tail integral of a Lévy process: intensity of simultaneous jumps smaller than a given value:

$$U(z_1,\dots,z_M)=\nu(\{x\in\mathbb{R}^M : x_1\leq z_1,\dots,x_M\leq z_M\}).$$

The marginal tail integral:

$$U_{i_1,\dots,i_m}(z_1,\dots,z_m)=\nu(\{x\in\mathbb{R}^M : x_{i_1}\leq z_1,\dots,x_{i_m}\leq z_m\}).$$

The Lévy copula (see Tankov '04, Kallsen & Tankov '06): links the multidimensional tail integral to the tail integrals of components:

$$U(z_1,\dots,z_M)=F(U_1(z_1),\dots,U_M(z_M)).$$

A parametric family of Lévy copulas

The Clayton family of Lévy copulas:

$$F(u_1, \ldots, u_M) = \left(u_1^{-\theta} + \cdots + u_M^{-\theta}\right)^{-1/\theta}.$$

- The limit $\theta \to +\infty$ corresponds to complete dependence and $\theta \to 0$ produces independent components.
- The Clayton family is margin-stable: if X has Clayton Lévy copula then all subsets of components also have Clayton Lévy copula.

Counting gap events

The process N_t counting gap events is a Lévy process with integer jump sizes $1, \dots, M$ and corresponding intensities $\lambda_1, \dots, \lambda_M$ given by

$$\lambda_m = \sum_{k=m}^{M} (-1)^{k-m} \sum_{i_1 < \dots < i_k} C_m^k U_{i_1,\dots,i_k}(\beta, \dots, \beta).$$

In the homogeneous case ($U_k(z) = U_1(z)$ for all k) with Clayton dependence,

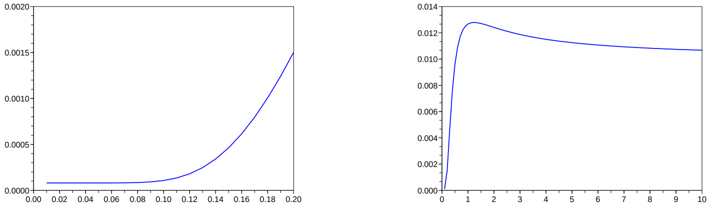
$$\lambda_m = U_1(\beta) C_m^M \sum_{j=0}^{M-m} \frac{(-1)^j C_j^{M-m}}{(m+j)^{1/\theta}} \xrightarrow{\theta \to \infty} \begin{cases} 0, & m < M \\ U_1(\beta), & m = M. \end{cases}$$

The pricing formula

Once the law of the number of gap events N_T is known, the expectation of any function of N_T is easy to evaluate.
Let f be the function from Crédit Suisse example.

$$\begin{aligned} E[f(N_T)] = e^{-\lambda T}\Big\{ & 1+\lambda_1 T+\frac{(\lambda_1 T)^2}{2}+\lambda_2 T \\ & +\frac{(\lambda_1 T)^3}{12}+\frac{\lambda_1\lambda_2 T^2}{2}+\frac{\lambda_3 T}{2}\Big\}, \\ \lambda = \lambda_1+\dots+\lambda_M. \end{aligned}$$

A numerical example



Expected loss of a multi-name gap option in the Credit Suisse example as a function of θ for 1% single name gap probability.

For Clayton copula, θ is linked to the probability p of having a gap in component A given a gap in component B: $\theta = -\frac{1}{\log_2 p}$; with $p = 25\%$, $\theta = \frac{1}{2}$.

Conclusions

> *…si Lehman Brothers avait établi ses calculs avec des processus de Lévy, la banque aurait distribué de plus faibles dividendes à ses actionnaires et se serait assuré plus de coussin de fonds propres…*
>
> *…If Lehman Brothers had used Lévy processes for its computations, this bank would have distributed less dividends to its shareholders and would have been left with a bigger cushion of capital…*

Le Nouvel Observateur, October 9, 2008.