# Arbitrage-free market models for liquid options

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# Introduction

# Option models

- **Problem:** Construct a dynamic model for
  - a stock $S$
  - a bank account ($\equiv 1$ in discounted units)
  - several liquidly traded European call options $C(K, T)$ on $S$

  in such a way that
  - the model is **arbitrage-free**
  - the model can **reproduce market prices** of $S$ and all $C(K, T)$
  - the **joint dynamics** of $S$ and all $C(K, T)$ are known explicitly

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- **Motivation – Ultimate goal:**
  Framework for arbitrage-free pricing and hedging of derivatives using stock, bank account, and liquid vanillas as hedge instruments

# Classical approach

Construction of arbitrage-free option models:

- Specify dynamics of $S$ under a pricing measure $Q$
- Define $C_t(K, T) := E_Q[(S_T - K)^+ | \mathcal{F}_t]$
- Calibrate to market prices

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**Drawbacks:**

- typically no closed form expressions for $C_t(K, T)$
- dynamics of $C_t(K, T)$ not explicitly known
- calibration may be difficult

$\rightarrow$ **need another approach...**

# ...market models!

**Basic idea:** Specify dynamics of all traded assets $S$ and $C(K, T)$ simultaneously via an SDE system

**Advantages:**

- option prices at time 0 are model input as initial conditions
- joint dynamics of $S$ and $C(K, T)$ known by construction
- capable of generating a rich class of price dynamics

# ...market models!

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**Challenge: Absence of arbitrage** must be guaranteed!

- **Static** arbitrage: Constraints on **state space** of $C(K, T)$
- **Dynamic** arbitrage: **Drift restrictions** in SDEs (cf. HJM!)

$\longrightarrow$ **How to deal with these constraints?**

# Arbitrage constraints

# Single option case

Consider market with bank account, stock $S$ and one call $C(K, T)$

- no **static** arbitrage: $(S_t - K)^+ \leq C_t \leq S_t$ and $C_T = (S_T - K)^+$
- no **dynamic** arbitrage: $S$ and $C$ (local) martingales under pricing measure $Q$

**Problem:** How to specify joint dynamics of $(S, C)$ fulfilling this?

# Stochastic implied volatility

**Idea** (→ Lyons 1997, Schönbucher 1999, Babbar 2001):
Use implied volatility $\hat{\sigma}$!

- **Stochastic implied volatility:**

$$dS_t = S_t \sigma_t dW_t^1,$$
$$d\hat{\sigma}_t = u_t dt + v_t dW_t$$

- $C_t = c_{\mathrm{BS}}\big(S_t, K, (T-t)\hat{\sigma}_t^2\big)$
- no static arbitrage ⇔ $\hat{\sigma} \in [0, \infty)$: nice **state space**
- no dynamic arbitrage ⇔ **drift restrictions** in SDE for $\hat{\sigma}$:

$$u_t = \mathcal{U}(S_t, \hat{\sigma}_t, v_t)$$

- **Existence** still tricky but feasible (explicit examples)

# Trouble with multi-option stochastic implied volatility...

- **Multi-option models:** static and dynamic conditions for each single option as before
- but **in addition**: static conditions across **different strikes and maturities**
  Condition across maturities: $T \mapsto C_t(K,T)$ increasing!
  Condition across strikes:
  - $K \mapsto C_t(K,T)$ convex
  - $-1 \leq \frac{\partial}{\partial K} C_t(K,T) \leq 0$
- Black-Scholes implied volatility does **not parametrize this in a simple way**!
  $\longrightarrow$ unclear how to ensure that SIV surface $\hat{\sigma}_t(K,T)$ stays in the arbitrage-free domain
- So what about existence of multi-option market models?

# $\longrightarrow$ New parametrizations!

**New parametrizations:**

- Schönbucher 1999: **forward implied volatilities** for multi-maturity models
- Schweizer and Wissel 2008: **local implied volatilities** for multi-strike models
- Carmona and Nadtochiy 2008: **local volatilities** for option surface models

# Construction of multi-option models

# Static arbitrage constraints

**The strike and maturity lattice**

Consider a market with bank account, stock $S$ and all calls $C(K_n, T_m)$ for $0 = K_0 < \cdots < K_{N+1}$ and $0 = T_0 < \cdots < T_M$. Write $C_n^m := C(K_n, T_m)$. Suppose $C_{N+1}^m = 0$ for all $m = 1, ..., M$.

**Static constraints:**

- $-1 \leq \frac{C_n^m - C_{n-1}^m}{K_n - K_{n-1}} \leq \frac{C_{n+1}^m - C_n^m}{K_{n+1} - K_n} \leq 0$
- $C_n^{m-1} \leq C_n^m$

We also have $C_0^m = S$ for all $m = 1, ..., M$.

# Local implied volatilities

## Definition (The new parametrization)

For statically arbitrage-free $C_n^m$ we define for each $\ell = 1, \ldots, M$ and $t \in [T_{\ell-1}, T_\ell]$ the **local implied volatilities** $x^m = (x_n^m)_{n=1,\ldots,N}$ by

$$x_n^m(t) = \sqrt{\frac{\frac{C_n^m(t) - C_n^{m-1}(t)}{T_m - T_{m-1}}}{\frac{K_n^2}{2}\left(\frac{C_{n+1}^m(t) - C_n^m(t)}{K_{n+1} - K_n} - \frac{C_n^m(t) - C_{n-1}^m(t)}{K_n - K_{n-1}}\right)\frac{2}{K_{n+1} - K_{n-1}}}} \quad (m > \ell).$$

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## Theorem (Prices in new parameters)

*There is a bijection between* **statically arbitrage-free** *option price models* $(C_n^m)$ *and all families* $(S, x_n^m)$ *of* **positive** *stock prices and local implied volatilities.*

# Model dynamics and option prices

The **model dynamics** is given iteratively on $t \in [T_{\ell-1}, T_\ell]$ by

$$S(t) = \sum_{j=0}^{N} \Phi\left(\frac{W^1(t) - W^1(T_{\ell-1}) + \Psi_j\Big(S(T_{\ell-1}), x^\ell(T_{\ell-1})\Big)}{\sqrt{T_\ell - t}}\right)(K_{j+1} - K_j)\,,$$

$$\frac{dx_n^m(t)}{x_n^m(t)} = u_n^m(t)dt + v_n^m(t)dW(t), \quad m = \ell+1, ..., M, \ n = 1, ..., N,$$

for $\ell = 1, ..., M$. The $\Psi_j$ are deterministic functions.

**Explicit formulas** for **call prices** $C^m = (C_1^m, ..., C_N^m)$ and their **dynamics** $dC_n^m(t) = ...dt + \lambda_n^m(t)dW(t)$:

$$C_n^\ell(t) = \sum_{j=n}^N \Phi\left(\frac{W^1(t)-W^1(T_{\ell-1})+\Psi_j\left(S(T_{\ell-1}), x^\ell(T_{\ell-1})\right)}{\sqrt{T_\ell - t}}\right)(K_{j+1} - K_j),$$

$$\lambda_n^\ell(t) = \sum_{j=n}^N \varphi\left(\frac{W^1(t)-W^1(T_{\ell-1})+\Psi_j\left(S(T_{\ell-1}), x^\ell(T_{\ell-1})\right)}{\sqrt{T_\ell - t}}\right)\frac{K_{j+1}-K_j}{\sqrt{T_\ell - t}},$$

$$C^m(t) = \left(I + \mathrm{Diag}\left(x^m(t)^2\right)B^m\right)^{-1}\left(C^{m-1}(t) + S(t)x_1^m(t)^2\beta^m e_1\right),$$

$$\lambda^m(t) = \left(I + \mathrm{Diag}\left(x^m(t)^2\right)B^m\right)^{-1}\left(\lambda^{m-1}(t) + \lambda_0^\ell(t)x_1^m(t)^2\beta_1^m e_1 + 2\,\mathrm{Diag}\left(v_1^m(t), ..., v_N^m(t)\right)\left(C^m(t) - C^{m-1}(t)\right)\right)$$

for $m > \ell$ and $t \in [T_{\ell-1}, T_\ell]$. The $B^m$ and $\beta^m$ are constants.

# Dynamic arbitrage constraints and existence result

**Absence of arbitrage** in the local implied volatility model can be characterized in terms of **drift restrictions**.

## Theorem (Drift restrictions)

*$S$ and all $C_n^m$ are loc. mart. under some $Q \approx P \quad \Leftrightarrow \quad \exists b$ such that*

$$\frac{dx_n^m(t)}{x_n^m(t)} = \left( \frac{3}{2} v_n^m(t) - \frac{\lambda_n^m(t) - \lambda_n^{m-1}(t)}{C_n^m(t) - C_n^{m-1}(t)} - b(t) \right) v_n^m(t)dt + v_n^m(t)dW(t)$$

*on $[0, T_{m-1}]$, for all $n$ and $m$.*

We can give **sufficient conditions** for the **existence of a market model** for local implied volatility with a **given volvol structure**.

**Theorem (Option model with prespecified volvol structure)**

*Suppose that for* $t \in [T_{\ell-1}, T_\ell]$ *and* $\ell = 1, ..., M$

$$v_n^{\ell+1}(t) = V\left(T_{\ell+1} - t, \frac{K_n}{S(t)}\right)\frac{1}{1+\epsilon(t)}$$

$$v_n^m(t) = V\left(T_m - t, \frac{K_n}{S(t)}\right) \qquad (m > \ell + 1)$$

*for a bounded* $V$ *and a suitable "small" control term* $\epsilon(t)$. *Then for each initial condition, the arbitrage-free local implied volatility SDE system has a unique solution* $(x_n^m)$. *By the previous result, the corresponding option model* $(S, C_n^m)$ *is arbitrage-free.*

# Outlook

# Open questions

- Extensions:
  - Existence results for full surface (all maturities and strikes)?
  - Can we include stochastic interest rates and dividends?
  - Can we go beyond the Brownian setting?
- Implementation:
  - Calibration?
  - Numerical solutions?
- Applications:
  - Pricing and hedging of derivatives with vanilla options?
- Structure of the models:
  - Markov property?
  - Finite-dimensional realizations?
  - Recalibration?

# Literature

- Single option case:
  Babbar (2001)
- Necessary conditions for option market models:
  Lyons (1997), Schönbucher (1999), Brace et al (2001), Ledoit et al (2002), Brace et al (2004), Jacod and Protter (2006), Carmona and Nadtochiy (2008)
- Smile consistent stock models (local volatility):
  Dupire (1994), Derman and Kani (1998), ...
- Sufficient conditions for existence of multi-option models:
  Schweizer and Wissel (2008), Wissel (2007)

Thank you for your attention!